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# On the stability of the dynamical system 'rigid body + inviscid fluid'

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In this paper we study a dynamical system consisting of a rigid body and an inviscid incompressible fluid. Two general configuraions of the system are considered: (*a*) a rigid body with a cavity completely filled with a fluid and (*b*) a rigid body surrounded by a fluid. In the first case the fluid is confined to an interior (for the body) domain and in the second case it occupies an exterior domain, which may, in turn, be bounded by some fixed rigid boundary or may extend to infinity. The aim of the paper is twofold: (i) to develop Arnold's technique for the system 'body + fluid' and (ii) to obtain sufficient conditions for the stability of steady states of the system. We first establish an energy-type variational principle for an arbitrary steady state of the system. Then we generalize this principle for states that are steady either in translationally moving in some fixed direction or rotating around some fixed axis coordinate system. The second variations of the corresponding functionals are calculated. The general results are applied to a number of particular stability problems. The first is the stability of a steady translational motion of a two-dimensional body in an irrotational flow. Here we have found that (for a quite wide class of bodies) the presence of non-zero circulation about the body does not affect its stability – a result that seems to be new. The second problem concerns the stability of a steady rotation of a force-free rigid body with a cavity containing an ideal fluid. Here we rediscover the stability criterion of Rumyantsev (see Moiseev & Rumyantsev 1965). The complementary problem – when a body is surrounded by a fluid and both body and fluid rotate with constant angular velocity around a fixed axis passing through the centre of mass of the body – is also considered and the corresponding sufficient conditions for stability are obtained.

## 1. Introduction

In this paper we present new general results concerning the stability of the mechanical system 'rigid body + inviscid fluid'. We consider two general configurations of the system: (*a*) a rigid body with a cavity filled with a fluid and (*b*) a rigid body surrounded by a fluid. In the first case the fluid is confined to an interior (for the body) domain. In the second case it occupies an exterior domain, which may, in turn, be bounded by some fixed rigid boundary or it may extend to infinity.

Studies of systems of both types have a long hystory, and the literature on the subject is too extensive be comprehensively reviewed here. We therefore restrict ourselves to only a few remarks on the previous results.

Since the fundamental work of Kelvin (see e.g. Lamb 1932, chapter XII, § 384) who experimentally discovered that a heavy top with a fluid-filled cavity is stable if the cavity is oblate and unstable if it is prolate, the stability of type (*a*) system has been studied by numerous authors. Most of the results that are available in the literature were obtained either for equilibria (states of rest) of the system or for a steady rotation of the system as a whole around some fixed axis. Extensive historical reviews on the subject may be found in the books by Moiseev & Rumyantsev (1965) and by Myshkis *et al.* (1987).

The flow associated with a rigid body moving in an inviscid incompressible fluid has also been intensively studied. In particular, the related mathematical theory of irrotational flow is quite well developed for both two- and three-dimensional problems (see e.g. Lamb 1932; Kelvin & Tait 1912; Batchelor 1967). The general situation of a rotational flow is much more complicated. Here most of the studies were concentrated on calculation of the force and moment exerted on a moving body (see e.g. the recent papers by Auton, Hunt & Prud'homme 1988 and by Howe 1995). The stability of a rigid body in a fluid flow has received much less attention. Apart from a number of classical results on the stability of a body in an uniform irrotational flow (see Lamb 1932; Kelvin & Tait 1912; Lyapunov 1954), there are only few (known to us) publications on the subject, e.g. Voinov & Petrov (1973, 1977) studied the stability of a rigid body in a non-uniform irrotational flow and in a flow with constant vorticity.

The aim of the present paper is twofold: first, to generalize Arnold's well-known technique (Arnold 1965*a,b*, 1966) so as to make it appropriate for the study of the system 'body + fluid' and, secondly, to investigate the stability of some steady states of both type (*a*) and type (*b*) systems. Accordingly, we first concentrate our efforts on the development of Arnolds' technique, and then we apply the general theory to obtain the stability results for some simple particular situations.

Our way of adapting Arnold's technique is quite straightforward but not trivial. Arnold's original technique was based on the underlying Hamiltonian structure of the Euler equations and, in particular, on the fact that the configuration space for an ideal incompressible fluid forms a group of volume-preserving diffeomorphisms of a (fixed) flow domain (Arnold 1966). For the system 'body + fluid', however, the flow domain changes with time, so that the configuration space does not form any group. In our treatment we avoid any explicit reference to the Hamiltonian structure of the system considered. Among other things, this may give us the possibility of generalizing the variational principles and the related stability results which we obtain here to systems that are not Hamiltonian. For instance, one can easily consider the energy dissipation in finite-dimensional degrees of freedom, i.e. add an extra term describing a velocity-dependent dissipative force to the equations of motion of the body.

Another important generalization of Arnold's technique concerns the situation when the basic state of the system is steady in some translationally moving (or steadily rotating round some fixed axis) frame of reference. This corresponds to a rigid body moving with constant velocity through a fluid that extends to infinity in some or all directions. We cannot just use the coordinate system fixed on the body because then the total energy of the system is infinite. Instead of this, we formulate the problem in the coordinate system which is fixed in space (so that the fluid is at rest at infinity) and construct the functional which is a certain linear combination of the total energy and the total momentum of the system and which has a critical point in the basic state considered. Such a variational principle is similar to that of Benjamin (1972).

The variational principles presented in the paper state that the total energy (or a linear combination of the energy and the momentum or the angular momentum) of the system has a critical point in a given basic state on the set of all 'isovortical flows'. The generalization of Arnold's original 'isovorticity' condition to the system 'body +

fluid' is given in a differential form first formulated for the case of fixed boundaries in Moffatt (1986) and Vladimirov (1987*a*). Arnold's general theory says (Arnold 1966) that the corresponding second variations are the invariants of the appropriate linearized equations provided that we identify the infinitesimal variations with small perturbations whose evolution is governed by these equations. The linear stability analysis reduces therefore to the study of the properties of the second variation.

The general theory is applied to three simple particular problems.

(1) The first one is the stability of a steady translational motion of a two-dimensional body in an irrotational flow. It is shown that such a motion of the body is stable if it moves in the direction corresponding to the maximum principal value of the added mass tensor. It is interesting that (for a quite wide class of bodies) the presence of non-zero circulation about the body does not affect its stability – a result that seems to be new.

(2) The second problem concerns the stability of a steady rotation of a force-free rigid body with a cavity containing an ideal fluid. Here we rediscover the stability criterion of Rumyantsev (see Moiseev & Rumyantsev 1965) which states that the system is stable provided it rotates around the principal axis of its moment of inertia tensor which corresponds to the maximum value of the moment of inertia.

(3) The third problem is, in some sense, complementary to the second one: a body and surrounding fluid are placed inside a fixed axisymmetric domain, and the basic steady state whose stability is studied is a steady rotation of both the body and the fluid around the axis of symmetry of the domain. The result is that this state is linearly stable if the density of the fluid is less than that of the body and the axis of rotation corresponds to the smallest moment of inertia of the body.

The results presented in this paper are interesting from two view-points: first, they lay the general basis for the subsequent stability analysis in various particular situations; secondly, these results represent a further development of Arnold's method in hydrodynamic stability theory.

The plan of the paper is as follows. In §2 we discuss the equations of motion for the dynamical system 'rigid body + fluid'. We assume that independent potential external forces are applied both to the body and to the fluid. The basic steady state whose stability is investigated is an equilibrium of the body in a steady rotational flow, so that in this state the force and the moment exerted on the body by the fluid are balanced by a suitable external force and torque applied to the body. In §3 we show that the energy of the system has a stationary value on the set of all the states of the system in which fluid flows are 'isovortical' to the basic steady flow. The corresponding second variation is calculated in §4.

In §§5 and 6 we deal with variational principles for those states of the system that are steady relative to some translationally moving or steadily rotating coordinate system (such states may correspond to a stationary, purely translational motion of a body in a channel filled with a fluid or to a steady rotation of a body in a fluid). We demonstrate that conserved functionals that are certain linear combinations of the total energy and the total momentum or angular momentum of the system have a critical point at the basic state provided that we take account only of 'isovortical' variations of the velocity field. The second variation of these functionals at critical points are calculated.

In §7 the two-dimensional problem is considered: the body is an infinite cylinder with an arbitrary cross-section moving perpendicularly to its axis, the flow is two-dimensional, i.e. it does not depend on the coordinate along the axis of a cylinder. In this case we establish a more general variational principle than for three-dimensional problems, namely, we waive the 'isovorticity condition' and admit arbitrary variations of the velocity field $\boldsymbol{u}$. The functional that has a critical point in the steady state is a certain linear combination of all constants of motion of the system. The second variation of this functional is calculated.

Finally, § 8 contains applications of the theory developed in the previous sections to three particular situations, namely, we study (i) the stability of a two-dimensional body moving with a constant velocity in an irrotational flow, (ii) the stability a force-free steady rotation of a rigid body with a cavity completely filled with a fluid and (iii) the stability of a body steadily rotating in a fluid.

## 2. Basic equations

Consider a dynamical system consisting of an incompressible, homogeneous and inviscid fluid and a rigid body. Let $\mathscr{D}$ be a domain in three-dimensional space that contains both the fluid and the rigid body, and let $\mathscr{D}_b(t)$ be a domain (inside $\mathscr{D}$, i.e. $\mathscr{D}_b(t) \subset \mathscr{D}$) occupied by the body. The domain $\mathscr{D}_f(t) = \mathscr{D} - \mathscr{D}_b(t)$ is completely filled with fluid; its boundary $\partial\mathscr{D}_f(t)$ consist of two parts: the inner boundary $\partial\mathscr{D}_b(t)$ representing the surface of the rigid body and the outer boundary $\partial\mathscr{D}$ which is fixed in the space.

In general, motion of the rigid body may or may not be restricted by some geometric constraints. The number of degrees of freedom is denoted by $N$ where necessarily $N \leqslant 6$. Motion of the body is described by its generalized coordinates $q_\alpha(t)$ and velocities $v_\alpha(t) = \dot{q}_\alpha \equiv \mathrm{d}q_\alpha/\mathrm{d}t$ ($\alpha = 1, ..., N$). Fluid motion is described by velocity field $u_i(\boldsymbol{x}, t)$ ($i = 1, 2, 3$) and the pressure field $p(\boldsymbol{x}, t)$, here $\boldsymbol{x} \equiv (x_1, x_2, x_3)$ are Cartesian coordinates. From here on we shall use two types of indices, Greek and italic. Greek indices take values from 1 to $N$ and correspond to finite-dimensional degrees of freedom of the system 'body + fluid', while italic indices take values from 1 to 3 and denote Cartesian components of vectors and tensors. In the rest of the paper the summation is implied over repeated both Greek and italic indices.

Forces that are external with respect to the system 'body + fluid' are applied both to the rigid body and to the fluid. The generalized force applied to the body is characterized by potential energy $\Pi(q_\alpha)$. The external force applied to the fluid is a body force (per unit mass) with a potential $\Phi(\boldsymbol{x})$.

The equations of motion for the fluid are the Euler equations:

$$\mathrm{D}\boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \quad \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \mathscr{D}_f, \tag{2.1}$$

where $\rho$ is the (constant) density of the fluid and $\mathrm{D} \equiv \partial/\partial t + \boldsymbol{u}\cdot\nabla$. The equation governing the evolution of vorticity field $\boldsymbol{\omega}(\boldsymbol{x}, t) \equiv \nabla\times\boldsymbol{u}$ follows from (2.1):

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times\left(\boldsymbol{u}\times\boldsymbol{\omega}\right). \tag{2.2}$$

Motion of the rigid body obeys the standard Lagrange equations of classical mechanics which may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial v_\alpha}\right] - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial \Pi}{\partial q_\alpha} + F_\alpha. \tag{2.3}$$

In (2.3), $T(q_\alpha, v_\alpha)$ is the kinetic energy of the body given by the equation

$$T = \tfrac{1}{2}Mw_iw_i + \tfrac{1}{2}I_{ik}\sigma_i\sigma_k, \tag{2.4}$$

where $I_{ik}$ is the moment of inertia tensor; the velocity of the centre of mass $\boldsymbol{w} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and the angular velocity $\boldsymbol{\sigma}$ are considered as functions of the generalized velocities $v_\alpha$ and coordinates $q_\alpha$ (if the constraints on the body are holonomic and time-independent, as we shall always assume here, then kinetic energy $T$ is a homogeneous quadratic form in the generalized velocities $v_\alpha$ (see e.g. Goldstein 1980)). $F_\alpha$ is given by the equation

$$F_\alpha = \int_{\partial\mathscr{D}_b} \left( \boldsymbol{n} \cdot \frac{\partial \boldsymbol{r}}{\partial q_\alpha} + \left[ (\boldsymbol{x} - \boldsymbol{r}) \times \boldsymbol{n} \right] \cdot \frac{\partial \boldsymbol{\sigma}}{\partial v_\alpha} \right) p \, \mathrm{d}s \tag{2.5}$$

and represents the $\alpha$-component of the generalized force exerted on the body by the fluid. In (2.5), $\boldsymbol{n}$ is the unit normal to the surface $\partial\mathscr{D}_b$; throughout the paper, for all boundaries the direction of $\boldsymbol{n}$ is always taken to be outward with respect to the fluid domain $\mathscr{D}_f$.

*Remark*. An instantaneous angular velocity $\boldsymbol{\sigma}$ of the rigid body is defined by the equation

$$\sigma_i \equiv -\tfrac{1}{2} e_{ijk} \frac{\mathrm{d}P_{jl}}{\mathrm{d}t} P_{kl}$$

where $e_{ijk}$ is the alternating tensor; $\left[P_{ik}\right]$ is an orthogonal matrix ($P_{il}P_{kl} = \delta_{ik}$) representing rotation from the axes $Ox_1x_2x_3$ of the coordinate system fixed in space to the axes $O'x'_1x'_2x'_3$ of the coordinate system fixed in the body (with the origin in its centre of mass), so that the position vector $\boldsymbol{x}$ of a point in the body relative to the space axes and the position vector $\boldsymbol{x}'$ of the same point measured by the body set of axes are related by the formula $x_i = r_i + P_{ij}x'_j$. The rotation matrix $\left[P_{ik}\right]$ is a function of the generalized coordinates $q_\alpha$; angular velocity $\boldsymbol{\sigma}$ can therefore be expressed in the form

$$\sigma_i = -\tfrac{1}{2} e_{ijk} \frac{\partial P_{jl}}{\partial q_\alpha} P_{kl} v_\alpha. \tag{2.6}$$

It is (2.6) that allows us to write the generalized force $F_\alpha$ in the form (2.5).

Boundary conditions for velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ are the usual ones of no normal flow through the rigid boundaries:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\mathscr{D}_b, \tag{2.7a}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = (\boldsymbol{w} + \boldsymbol{\sigma} \times (\boldsymbol{x} - \boldsymbol{r})) \cdot \boldsymbol{n} \quad \text{on} \quad \partial\mathscr{D}_b. \tag{2.7b}$$

Equations (2.1), (2.3) with boundary conditions (2.7) give us the complete set of equations governing the motion of the system 'body + fluid'.

The conserved total energy of the system is given by

$$E = E_f + E_b = \text{const}, \quad E_b \equiv T + \Pi, \tag{2.8a}$$

$$E_f \equiv \int_{\mathscr{D}_f} \rho \left( \frac{u_i u_i}{2} + \Phi \right) \mathrm{d}\tau, \quad \mathrm{d}\tau \equiv \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3. \tag{2.8b}$$

Steady solutions of the problem (2.1), (2.3), (2.7) given by

$$v_\alpha = 0, \;\; q_\alpha = Q_\alpha, \;\; \boldsymbol{r} = \boldsymbol{R} = 0, \;\; \boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}), \;\; p = P(\boldsymbol{x}), \;\; \boldsymbol{\omega} = \boldsymbol{\Omega}(\boldsymbol{x}), \tag{2.9a}$$

$$\boldsymbol{w} = \boldsymbol{W} = 0, \;\; \boldsymbol{\sigma} = \boldsymbol{\Sigma} = 0, \;\; P_{ij} = P_{0ij} = \delta_{ij} \tag{2.9b}$$

satisfy the equations

$$\boldsymbol{\Omega} \times \boldsymbol{U} = -\nabla H, \quad H \equiv P/\rho + \Phi + \tfrac{1}{2}\boldsymbol{U}^2, \quad \nabla \cdot \boldsymbol{U} = 0 \quad \text{in} \;\; \mathscr{D}_{f0}; \tag{2.10}$$

$$-\frac{\partial \Pi}{\partial Q_\alpha} + \int_{\partial \mathscr{D}_{b0}} \left( \boldsymbol{n} \cdot \frac{\partial \boldsymbol{R}}{\partial Q_\alpha} + \left[ (\boldsymbol{x} - \boldsymbol{R}) \times \boldsymbol{n} \right] \cdot \frac{\partial \boldsymbol{\Sigma}}{\partial V_\alpha} \right) P \, \mathrm{d}s = 0; \tag{2.11}$$

and boundary conditions

$$\boldsymbol{U} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial \mathscr{D} \quad \text{and on} \quad \partial \mathscr{D}_{b0}. \tag{2.12}$$

In (2.10)–(2.12) boundary $\partial \mathscr{D}_{b0}$ corresponds to the equilibrium position of the rigid body. In obtaining (2.11) we used the fact that, according to (2.4), (2.9), $\partial T / \partial Q_\alpha = 0$. Steady solution (2.9) represents an equilibrium of the body in a steady rotational flow.

## 3. Variational principle

In this section, following the procedure of Vladimirov (1987*a*), who proposed a somewhat simpler and more descriptive (from a physical view-point) form of the original variational principle of Arnold (1965*b*), we shall show that the total energy of the dynamical system ‘body + fluid’ has a stationary value at the steady solution (2.9) on the set of all possible fluid flows that are ‘isovortical’ to the basic flow (see Arnold 1965*b*). The ‘isovorticity’ condition means that we admit only such variations of the velocity field $\boldsymbol{u}$ that preserve the velocity circulation over any material contour.

To formulate the variational principle we introduce the family of transformations

$$\tilde{x}_i = \tilde{x}_i(\boldsymbol{x}, \epsilon), \tag{3.1a}$$

$$\tilde{q}_\alpha = \tilde{q}_\alpha(\epsilon), \tag{3.1b}$$

depending on a parameter $\epsilon \geqslant 0$ where the functions $\tilde{x}_i(\boldsymbol{x}, \epsilon)$ and $\tilde{q}_\alpha(\epsilon)$ are twice differentiable with respect to $\epsilon$ and the value $\epsilon = 0$ corresponds to the steady solution (2.9):

$$\tilde{x}_i(\boldsymbol{x}, 0) = x_i, \quad \tilde{q}_\alpha(0) = Q_\alpha. \tag{3.2}$$

The transformations defined by (3.1)–(3.2) can be interpreted as a ‘virtual motion’ of the system ‘body + fluid’ where $\epsilon$ plays the role of a ‘virtual time’, $\tilde{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)$ is the position vector at the moment of ‘time’ $\epsilon$ of a fluid particle whose position at the initial instant $\epsilon = 0$ was $\boldsymbol{x}$ (in other words, $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathscr{D}_{f0}$) serves as a label to identify the fluid particle, while $\tilde{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)$ represents its trajectory) and where the functions $\tilde{q}_\alpha(\epsilon)$ determine the position and the orientation of the rigid body at the moment of ‘time’ $\epsilon$. In such a motion, the domain $\mathscr{D}_{f0} = \tilde{\mathscr{D}}_f(0)$ evolves to a new one $\tilde{\mathscr{D}}_f(\epsilon)$ which is completely determined by the position and the orientation of the rigid body, i.e. by the generalized coordinates $\tilde{q}_\alpha(\epsilon)$.

Functions $\tilde{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)$, $\tilde{q}_\alpha(\epsilon)$ are specified through yet another set of functions $\boldsymbol{f}(\tilde{\boldsymbol{x}}, \epsilon)$, $h_\alpha(\epsilon)$ by the equations

$$\mathrm{d}\tilde{\boldsymbol{x}} / \mathrm{d}\epsilon = \boldsymbol{f}(\tilde{\boldsymbol{x}}, \epsilon), \tag{3.3a}$$

$$\mathrm{d}\tilde{q}_\alpha / \mathrm{d}\epsilon = h_\alpha(\epsilon), \tag{3.3b}$$

where $h_\alpha(\epsilon)$ are arbitrary differentiable functions, while $\boldsymbol{f}(\tilde{\boldsymbol{x}}, \epsilon)$ is an arbitrary vector field differentiable with respect to $\epsilon$ and satisfying the conditions

$$\tilde{\nabla} \cdot \boldsymbol{f} = 0 \quad \text{in} \quad \tilde{\mathscr{D}}_f(\epsilon), \tag{3.4a}$$

$$\boldsymbol{f} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial \tilde{\mathscr{D}}. \tag{3.4b}$$

$$\boldsymbol{f} \cdot \boldsymbol{n} = \left[ \tilde{\boldsymbol{r}}_\epsilon + \tilde{\boldsymbol{\varphi}}_\epsilon \times \left( \tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}} \right) \right] \cdot \boldsymbol{n} \quad \text{on} \quad \partial \tilde{\mathscr{D}}_b(\epsilon). \tag{3.4c}$$

In (3.4),

$$\tilde{\boldsymbol{r}}_\epsilon \equiv \frac{\partial \tilde{\boldsymbol{r}}}{\partial \tilde{q}_\alpha} h_\alpha, \quad \tilde{\varphi}_{i\epsilon} \equiv -\tfrac{1}{2} e_{ijk} \frac{\partial \tilde{P}_{jl}}{\partial \tilde{q}_\alpha} \tilde{P}_{kl} h_\alpha. \tag{3.5}$$

In terms of 'virtual motions' the functions $\boldsymbol{f}(\tilde{\boldsymbol{x}}, \epsilon)$ and $h_\alpha(\epsilon)$ entering (3.3) have a natural interpretation as the 'virtual velocities' of the fluid and the rigid body. The conditions (3.4) mean that in 'virtual motion' the fluid remains incompressible and that there is no fluid 'flow' through the rigid boundaries.

The actual velocity field of the fluid and the actual generalized velocities of the rigid body in 'virtual motion' are described by twice differentiable (with respect to $\epsilon$) functions $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)$ and $\tilde{v}_\alpha(\epsilon)$ such that the value $\epsilon = 0$ corresponds to the steady state (2.9):

$$\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)\big|_{\epsilon=0} = \boldsymbol{U}(\tilde{\boldsymbol{x}}), \quad \tilde{v}_\alpha(\epsilon)\big|_{\epsilon=0} = 0. \tag{3.6}$$

In addition, the field $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)$ satisfies the conditions

$$\tilde{\nabla} \cdot \tilde{\boldsymbol{u}} = 0 \quad \text{in} \quad \tilde{\mathscr{D}}_f, \tag{3.7a}$$

$$\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\tilde{\mathscr{D}}, \tag{3.7b}$$

$$\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = \left(\tilde{\boldsymbol{w}} + \tilde{\boldsymbol{\sigma}} \times (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}})\right) \cdot \boldsymbol{n} \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(\epsilon), \tag{3.7c}$$

where, as before, $\tilde{\boldsymbol{w}}$, $\tilde{\boldsymbol{\sigma}}$ are considered as functions of $\tilde{v}_\alpha(\epsilon)$ and $\tilde{q}_\alpha(\epsilon)$. The evolution with the 'time' $\epsilon$ of the generalized velocities $\tilde{v}_\alpha(\epsilon)$ is prescribed by the equation

$$\mathrm{d}\tilde{v}_\alpha / \mathrm{d}\epsilon = g_\alpha(\epsilon) \tag{3.8}$$

with some differentiable function $g_\alpha(\epsilon)$. Note that the functions $g_\alpha(\epsilon)$ and $h_\alpha(\epsilon)$ which determine the evolution in the 'virtual motion' of the generalized velocities and coordinates are both arbitrary, so that $\tilde{v}_\alpha(\epsilon)$ and $\tilde{q}_\alpha(\epsilon)$ vary independently.

The evolution of the field $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)$ is defined through the evolution of vorticity $\tilde{\boldsymbol{\omega}}(\tilde{\boldsymbol{x}}, \epsilon) \equiv \tilde{\nabla} \times \tilde{\boldsymbol{u}}$ by the equation

$$\tilde{\boldsymbol{\omega}}_\epsilon = \tilde{\nabla} \times \left(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}\right), \tag{3.9}$$

where subscript $\epsilon$ denotes partial derivative with respect to $\epsilon$. Equation (3.9) means that the vorticity field $\tilde{\boldsymbol{\omega}}$ is considered as a passive vector advected by the 'virtual flow' rather than as a field related to the 'virtual velocity' $\boldsymbol{f}$ by the curl-operator; in other words, the evolution of $\boldsymbol{\omega}$ is the same as that of a material line element $\delta \boldsymbol{l}$ or as that of a frozen-in magnetic field in ideal MHD. Yet another meaning of equation (3.9) is that the circulation of the velocity field $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)$ round any closed material curve is conserved in the 'virtual motion', this, in turn, implies that (3.9) is equivalent to the original isovorticity condition of Arnold (1965*b*).

On integrating (3.9) we obtain

$$\tilde{\boldsymbol{u}}_\epsilon = \boldsymbol{f} \times \tilde{\boldsymbol{\omega}} + \tilde{\nabla}\alpha \tag{3.10}$$

with a certain function $\alpha(\tilde{\boldsymbol{x}}, \epsilon)$ which, as follows from (3.7), (3.10), can be found by solving the problem

$$\tilde{\nabla}^2 \alpha = -\tilde{\nabla} \cdot \left(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}\right) \quad \text{in} \quad \tilde{\mathscr{D}}_f, \tag{3.11a}$$

$$\boldsymbol{n} \cdot \tilde{\nabla}\alpha = -\boldsymbol{n} \cdot \left(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}\right) \quad \text{on} \quad \partial\tilde{\mathscr{D}}, \tag{3.11b}$$

$$\boldsymbol{n} \cdot \tilde{\nabla}\alpha = \tilde{\boldsymbol{u}}_\epsilon \cdot \boldsymbol{n} - \boldsymbol{n} \cdot \left(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}\right) \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(\epsilon), \tag{3.11c}$$

where $\tilde{\boldsymbol{u}}_\epsilon \cdot \boldsymbol{n}$ can be obtained by differentiating the condition (3.7*c*) with respect to $\epsilon$. Function $\alpha(\tilde{\boldsymbol{x}}, \epsilon)$ is uniquely determined by (3.11*a*) with boundary conditions (3.11*b*,*c*) provided that the domain $\tilde{\mathscr{D}}_f(\epsilon)$ is singly-connected.

*Remark.* Though (3.10) also could be used as a primary condition for defining the evolution of the field $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)$, from a view-point of physical interpretation (3.9) seems preferable.

Assuming that $\epsilon$ is small we define the first and second variations of the velocity field of the fluid $\boldsymbol{u}$ and the generalized velocities and coordinates of the rigid body $v_\alpha$, $q_\alpha$ as follows:

$$\left.\begin{aligned} \delta \boldsymbol{x} &\equiv \boldsymbol{f}(\boldsymbol{x},0)\,\epsilon, \quad \delta \boldsymbol{u}(\boldsymbol{x}) \equiv \tilde{\boldsymbol{u}}_\epsilon(\boldsymbol{x},0)\,\epsilon, \quad \delta^2 \boldsymbol{u}(\boldsymbol{x}) \equiv \tfrac{1}{2}\tilde{\boldsymbol{u}}_{\epsilon\epsilon}(\boldsymbol{x},0)\,\epsilon^2, \\ \delta v_\alpha &\equiv v_{\alpha\epsilon}(0)\,\epsilon = g_\alpha(0)\epsilon, \quad \delta^2 v_\alpha \equiv \tfrac{1}{2}\tilde{v}_{\alpha\epsilon\epsilon}(0)\,\epsilon^2 = \tfrac{1}{2}g_{\alpha\epsilon}(0)\,\epsilon^2, \\ \delta q_\alpha &\equiv \tilde{q}_{\alpha\epsilon}(0)\,\epsilon = h_\alpha(0)\,\epsilon, \quad \delta^2 q_\alpha \equiv \tfrac{1}{2}\tilde{q}_{\alpha\epsilon\epsilon}(0)\,\epsilon^2 = \tfrac{1}{2}h_{\alpha\epsilon}(0)\,\epsilon^2. \end{aligned}\right\} \tag{3.12}$$

In (3.12), $\delta \boldsymbol{x}$ is the Lagrangian displacement of the fluid element whose position at time $t$ in the undisturbed flow was $\boldsymbol{x}$. The first and second variations of the energy (2.8) considered as a functional of $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)$, $\tilde{v}_\alpha(\epsilon)$, $\tilde{q}_\alpha(\epsilon)$ are, by definition,

$$\delta E \equiv \mathrm{d}E/\mathrm{d}\epsilon\big|_{\epsilon=0}\epsilon, \qquad \delta^2 E \equiv \tfrac{1}{2}\mathrm{d}^2 E/\mathrm{d}\epsilon^2\big|_{\epsilon=0}\epsilon^2. \tag{3.13}$$

The first variation of $E$ is

$$\delta E = \delta E_f + \delta E_b.$$

From (2.4) it follows that

$$\delta E_b = M W_i \delta w_i + \tfrac{1}{2}\delta I_{ik}\Sigma_i\Sigma_k + I_{ik}\Sigma_i\delta\sigma_k + \frac{\partial \Pi}{\partial Q_\alpha}\delta q_\alpha \tag{3.14}$$

where

$$\delta \boldsymbol{w} = \frac{\partial \boldsymbol{W}}{\partial Q_\alpha}\delta q_\alpha + \frac{\partial \boldsymbol{W}}{\partial V_\alpha}\delta v_\alpha, \quad \delta \boldsymbol{\sigma} = \frac{\partial \boldsymbol{\Sigma}}{\partial Q_\alpha}\delta q_\alpha + \frac{\partial \boldsymbol{\Sigma}}{\partial V_\alpha}\delta v_\alpha, \quad \delta I_{ik} = \frac{\partial I_{ik}}{\partial Q_\alpha}\delta q_\alpha.$$

Since in the basic state of the system $\boldsymbol{W} = \boldsymbol{\Sigma} = 0$, we obtain

$$\delta E_b = \frac{\partial \Pi}{\partial Q_\alpha}\delta q_\alpha. \tag{3.15}$$

To calculate $\delta E_f$ we first note that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_{\tilde{\mathscr{D}}_f(\epsilon)} F(\tilde{\boldsymbol{x}}, \epsilon)\mathrm{d}\tau = \int_{\tilde{\mathscr{D}}_f(\epsilon)} F_\epsilon \mathrm{d}\tau + \int_{\partial\tilde{\mathscr{D}}_b(\epsilon)} F\big(\boldsymbol{f}\cdot\boldsymbol{n}\big)\mathrm{d}s$$

for any function $F(\tilde{\boldsymbol{x}}, \epsilon)$ (this follows from the formula for the rate of change of material volume integral; see e.g. Batchelor 1967). With help of this we find

$$\frac{\mathrm{d}E_f(0)}{\mathrm{d}\epsilon} = \int_{\tilde{\mathscr{D}}_f(0)} \rho \boldsymbol{U}\cdot\tilde{\boldsymbol{u}}_\epsilon d\tau + \int_{\partial\tilde{\mathscr{D}}_b(0)} \rho\left(\tfrac{1}{2}\boldsymbol{U}^2 + \Phi\right)\big(\boldsymbol{f}\cdot\boldsymbol{n}\big)\mathrm{d}s.$$

On substituting (3.10) in this equation, we obtain

$$\frac{\mathrm{d}E_f(0)}{\mathrm{d}\epsilon} = \int_{\tilde{\mathscr{D}}_f(0)} \rho\{\boldsymbol{f}\cdot(\boldsymbol{\Omega}\times\boldsymbol{U}) + \boldsymbol{U}\cdot\tilde{\nabla}\alpha\}\mathrm{d}\tau + \int_{\partial\tilde{\mathscr{D}}_b(0)} \rho\left(\tfrac{1}{2}\boldsymbol{U}^2 + \Phi\right)\big(\boldsymbol{f}\cdot\boldsymbol{n}\big)\mathrm{d}s.$$

By using (2.10), Green's theorem and the boundary condition (3.4*c*), this can be transformed to

$$\frac{\mathrm{d}E_f(0)}{\mathrm{d}\epsilon} = -\int_{\partial\tilde{\mathscr{D}}_b(0)} P[\tilde{\boldsymbol{r}}_\epsilon + \tilde{\boldsymbol{\varphi}}_\epsilon \times (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}})] \cdot \boldsymbol{n}\,\mathrm{d}s. \tag{3.16}$$

Finally, from (3.15), (3.16) it follows that

$$\delta E = \frac{\partial \Pi}{\partial Q_\alpha}\delta q_\alpha - \int_{\partial\mathscr{D}_b(0)} P(\delta\boldsymbol{r} + \delta\boldsymbol{\varphi} \times (\boldsymbol{x} - \boldsymbol{r})) \cdot \boldsymbol{n}\,\mathrm{d}s. \tag{3.17}$$

In (3.17) the 'tildes' have been dropped (in accordance with the condition $\tilde{\boldsymbol{x}}(\boldsymbol{x}, 0) = \boldsymbol{x}$). The comparison of (3.17) with (2.11) then shows that $\delta E = 0$. Thus, we have shown that the energy of the system 'body + fluid' has a stationary value at any steady solution of the form (2.9) provided that we take account only of 'isovortical' fluid flows. This result is a natural generalization of Arnold's well-known variational principle (Arnold 1965*b*).

## 4. The second variation

Let us now calculate the second variation of the energy (2.8) at the stationary point. We have

$$\delta^2 E = \delta^2 E_f + \delta^2 E_b. \tag{4.1}$$

Consider first the part of $\delta^2 E$ corresponding to the rigid body degrees of freedom. From (2.4), (2.9) it follows that

$$\begin{aligned}\delta^2 E_b = {} & \tfrac{1}{2}M\delta w_i \delta w_i + M W_i \delta^2 w_i \\ & + \tfrac{1}{2}\delta^2 I_{ik}\Sigma_i\Sigma_k + \delta I_{ik}\Sigma_i\delta\sigma_k + \tfrac{1}{2}I_{ik}\delta\sigma_i\delta\sigma_k + I_{ik}\Sigma_i\delta^2\sigma_k + \delta^2\Pi,\end{aligned} \tag{4.2}$$

where

$$\delta^2\Pi = \tfrac{1}{2}\frac{\partial^2\Pi}{\partial Q_\alpha \partial Q_\beta}\delta q_\alpha \delta q_\beta + \frac{\partial \Pi}{\partial Q_\alpha}\delta^2 q_\alpha.$$

Since in the basic state (2.9) $\boldsymbol{W} = \boldsymbol{\Sigma} = 0$, (4.2) reduces to

$$\delta^2 E_b = \tfrac{1}{2}M\delta w_i\delta w_i + \tfrac{1}{2}I_{ik}\delta\sigma_i\delta\sigma_k + \tfrac{1}{2}\frac{\partial^2\Pi}{\partial Q_\alpha \partial Q_\beta}\delta q_\alpha\delta q_\beta + \frac{\partial\Pi}{\partial Q_\alpha}\delta^2 q_\alpha. \tag{4.3}$$

Consider now $\delta^2 E_f$. It can be shown (see the Appendix) that for any function $F(\tilde{\boldsymbol{x}}, \epsilon)$ the following equation holds:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\int_{\tilde{\mathscr{D}}_f} F(\tilde{\boldsymbol{x}}, \epsilon)\mathrm{d}\tau = {} & \int_{\tilde{\mathscr{D}}_f} F_{\epsilon\epsilon}\mathrm{d}\tau + \int_{\partial\tilde{\mathscr{D}}_b} (2F_\epsilon + \boldsymbol{y}_\epsilon \cdot \tilde{\nabla} F)(\boldsymbol{y}_\epsilon \cdot \boldsymbol{n})\mathrm{d}s \\ & + \int_{\partial\tilde{\mathscr{D}}_b} F(\tilde{\boldsymbol{r}}_\epsilon \times \tilde{\boldsymbol{\varphi}}_\epsilon) \cdot \boldsymbol{n}\,\mathrm{d}s + \int_{\partial\tilde{\mathscr{D}}_b} F\left[\tilde{\boldsymbol{r}}_{\epsilon\epsilon} + \tilde{\boldsymbol{\varphi}}_{\epsilon\epsilon} \times (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}})\right] \cdot \boldsymbol{n}\,\mathrm{d}s\end{aligned} \tag{4.4}$$

where

$$\boldsymbol{y}_\epsilon \equiv \tilde{\boldsymbol{r}}_\epsilon + \tilde{\boldsymbol{\varphi}}_\epsilon \times (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}}). \tag{4.5}$$

Applying this formula to $\mathrm{d}^2 E_f/\mathrm{d}\epsilon^2$, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2 E_f(0)}{\mathrm{d}\epsilon^2} = {} & \int_{\tilde{\mathscr{D}}_f(0)} \rho(\tilde{\boldsymbol{u}}_\epsilon^2 + \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{\epsilon\epsilon})\mathrm{d}\tau + \int_{\partial\tilde{\mathscr{D}}_b(0)} \rho(2\boldsymbol{U} \cdot \boldsymbol{u}_\epsilon + \boldsymbol{y}_\epsilon \cdot \tilde{\nabla} G)(\boldsymbol{y}_\epsilon \cdot \boldsymbol{n})\mathrm{d}s \\ & + \int_{\partial\tilde{\mathscr{D}}_b(0)} \rho G(\tilde{\boldsymbol{r}}_\epsilon \times \tilde{\boldsymbol{\varphi}}_\epsilon + \tilde{\boldsymbol{r}}_{\epsilon\epsilon} + \tilde{\boldsymbol{\varphi}}_{\epsilon\epsilon} \times \tilde{\boldsymbol{x}}) \cdot \boldsymbol{n}\,\mathrm{d}s,\end{aligned} \tag{4.6}$$

where

$$G \equiv \tfrac{1}{2}\boldsymbol{U}^2 + \Phi. \tag{4.7}$$

From (3.10) we have

$$\tilde{\boldsymbol{u}}_{\epsilon\epsilon}|_{\epsilon=0} = \boldsymbol{f}_\epsilon \times \boldsymbol{\Omega} + \boldsymbol{f} \times \tilde{\boldsymbol{\omega}}_\epsilon + \tilde{\nabla}\alpha_\epsilon.$$

On substituting this in the volume integral on the right-hand side of (4.6) and using (2.10), we obtain

$$\int_{\tilde{\mathscr{D}}_f(0)} \rho \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{\epsilon\epsilon} \mathrm{d}\tau = \int_{\tilde{\mathscr{D}}_f(0)} \rho(-\boldsymbol{f}_\epsilon \cdot \tilde{\nabla} H + \boldsymbol{U} \cdot \big(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}_\epsilon\big) + \boldsymbol{U} \cdot \tilde{\nabla}\alpha_\epsilon)\mathrm{d}\tau,$$

whence, after using Green's formula and the conditions (2.12) and (3.4),

$$\int_{\tilde{\mathscr{D}}_f(0)} \rho \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{\epsilon\epsilon} \mathrm{d}\tau = \int_{\tilde{\mathscr{D}}_f(0)} \rho \boldsymbol{U} \cdot \big(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}_\epsilon\big)\mathrm{d}\tau + \int_{\partial\tilde{\mathscr{D}}_b(0)} \rho H \big(\boldsymbol{f}_\epsilon \cdot \boldsymbol{n}\big)\mathrm{d}s. \tag{4.8}$$

Boundary conditions for the field $\boldsymbol{f}_\epsilon$ are given by (A3) in the Appendix. After using these boundary conditions in (4.8), substituting the resulting formula in (4.6) and after some further manipulations, one can arrive at the formula

$$\begin{aligned} \frac{\mathrm{d}^2 E_f(0)}{\mathrm{d}\epsilon^2} &= \int_{\tilde{\mathscr{D}}_f(0)} \rho(\tilde{\boldsymbol{u}}_\epsilon^2 + \boldsymbol{U} \cdot \big(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}_\epsilon\big))\mathrm{d}\tau \\ &\quad - \int_{\partial\tilde{\mathscr{D}}_b(0)} P(\tilde{\boldsymbol{r}}_\epsilon \times \tilde{\boldsymbol{\varphi}}_\epsilon + \tilde{\boldsymbol{r}}_{\epsilon\epsilon} + \tilde{\boldsymbol{\varphi}}_{\epsilon\epsilon} \times \tilde{\boldsymbol{x}}) \cdot \boldsymbol{n}\,\mathrm{d}s \\ &\quad + \int_{\partial\tilde{\mathscr{D}}_b(0)} (2\rho \boldsymbol{U} \cdot \boldsymbol{u}_\epsilon - \boldsymbol{y}_\epsilon \cdot \tilde{\nabla} P + \rho \boldsymbol{f} \cdot \tilde{\nabla} H)\big(\boldsymbol{y}_\epsilon \cdot \boldsymbol{n}\big)\mathrm{d}s. \end{aligned} \tag{4.9}$$

Using (2.6), (3.5), it can be shown that

$$\left.\begin{aligned} \left.\frac{\mathrm{d}^2 \tilde{\boldsymbol{r}}}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} &= \frac{\partial \boldsymbol{R}}{\partial Q_\alpha}\frac{\partial^2 \tilde{q}_\alpha}{\partial \epsilon^2} + \frac{\partial^2 \boldsymbol{R}}{\partial Q_\alpha \partial Q_\beta}\frac{\partial \tilde{q}_\alpha}{\partial \epsilon}\frac{\partial \tilde{q}_\beta}{\partial \epsilon}, \\ \left.\frac{\mathrm{d}^2 \tilde{\boldsymbol{\varphi}}}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} &= \frac{\partial \boldsymbol{\Sigma}}{\partial V_\alpha}\frac{\partial^2 \tilde{q}_\alpha}{\partial \epsilon^2} + \frac{\partial^2 \boldsymbol{\Sigma}}{\partial V_\alpha \partial Q_\beta}\frac{\partial \tilde{q}_\alpha}{\partial \epsilon}\frac{\partial \tilde{q}_\beta}{\partial \epsilon}. \end{aligned}\right\} \tag{4.10}$$

Finally, collecting (4.3), (4.9) and taking account of (4.10), we obtain

$$\delta^2 E = \delta^2 E_A + \delta^2 E_c + \delta^2 E_b, \tag{4.11a}$$

$$\delta^2 E_A \equiv \frac{1}{2}\int_{\mathscr{D}_{f0}} \rho\{\big(\delta \boldsymbol{u}\big)^2 + \boldsymbol{U} \cdot \big(\delta \boldsymbol{x} \times \delta \boldsymbol{\omega}\big)\}\mathrm{d}\tau, \tag{4.11b}$$

$$\begin{aligned} \delta^2 E_c &\equiv \frac{1}{2}\int_{\partial\mathscr{D}_{b0}} \{2\rho \boldsymbol{U} \cdot \delta \boldsymbol{u} - \delta \boldsymbol{y} \cdot \nabla P\}\big(\delta \boldsymbol{y} \cdot \boldsymbol{n}\big)\mathrm{d}s + \frac{1}{2}\int_{\partial\mathscr{D}_{b0}} \rho\big(\delta \boldsymbol{y} \cdot \boldsymbol{n}\big)(\delta \boldsymbol{x} \cdot \nabla H)\mathrm{d}s \\ &\quad - \frac{1}{2}\int_{\partial\mathscr{D}_{b0}} P\{\boldsymbol{n} \cdot \big[\delta \boldsymbol{r} \times \delta \boldsymbol{\varphi}\big] + A_{\alpha\beta}\delta q_\alpha \delta q_\beta + B_{\alpha\beta}\delta q_\alpha \delta q_\beta\}\mathrm{d}s, \end{aligned} \tag{4.11c}$$

$$\delta^2 E_b \equiv \tfrac{1}{2} M \delta w_i \delta w_i + \tfrac{1}{2} I_{ik} \delta \sigma_i \delta \sigma_k + \tfrac{1}{2}\frac{\partial^2 \Pi}{\partial Q_\alpha \partial Q_\beta}\delta q_\alpha \delta q_\beta, \tag{4.11d}$$

where $\delta\boldsymbol{y} \equiv \delta\boldsymbol{r} + \delta\boldsymbol{\varphi} \times \boldsymbol{x}$ is the displacement of a point on the body surface and where

$$A_{\alpha\beta} \equiv \boldsymbol{n} \cdot \boldsymbol{R}_{\alpha\beta}, \quad B_{\alpha\beta} \equiv \boldsymbol{n} \cdot \left[\boldsymbol{\Sigma}_{\alpha\beta} \times \boldsymbol{x}\right], \tag{4.12a}$$

$$\boldsymbol{R}_{\alpha\beta} \equiv \frac{\partial^2 \boldsymbol{R}}{\partial Q_\alpha \partial Q_\beta}, \quad \boldsymbol{\Sigma}_{\alpha\beta} \equiv \frac{\partial^2 \boldsymbol{\Sigma}}{\partial V_\alpha \partial Q_\beta}. \tag{4.12b}$$

In (4.11) $\delta^2 E_A$ is precisely the second variation of the energy of the fluid in the fixed domain $\mathscr{D}_{f0}$ (Arnold 1965*b*); $\delta^2 E_b$ involves only the variations of the generalized coordinates and velocities of the rigid body; $\delta^2 E_c$ depends on the variations of fluid variables and rigid body variables, so it may be interpreted as that part of $\delta^2 E$ appearing due to interaction between the body and the flow.

The remarkable fact about the second variation $\delta^2 E$ is that if we consider the variations $\delta\boldsymbol{x}$, $\delta\boldsymbol{u}$ and $\delta q_\alpha$ as the infinitesimal disturbances, whose evolution is governed by appropriate linearized equations, then $\delta^2 E$ is an invariant of these equations (Arnold 1965*a, b*, 1966; see also Holm *et al.* 1985; Vladimirov 1987*b*). From this fact it immediately follows that the basic state (2.9) is linearly stable provided that $\delta^2 E$ is positive definite. The linear stability problem thus reduces to the analysis of the second variation.

### 4.1. *Euler angles*

Now consider the situation when no constraints are imposed on the motion of the rigid body. In this case it is natural to take as the generalized coordinates three Cartesian components of the radius-vector of the centre of mass of the body and three Euler angles $\phi$, $\theta$, $\psi$ that characterize the orientation of the body in space. In defining the Euler angles we shall use the $xyz$-convention (as described in Goldstein 1980), so that they are specified by an initial rotation about the original $z$-axis through an angle $\phi$, a second rotation about the intermediate $y$-axis, and a third rotation about the final $x$-axis through an angle $\psi$. With this choice the components of the angular velocity $\boldsymbol{\sigma}$ along the space axis are (see Goldstein 1980, p. 610)

$$\left.\begin{aligned} \sigma_1 &= \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi, \\ \sigma_2 &= \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi, \\ \sigma_3 &= \dot{\phi} - \dot{\psi}\sin\theta. \end{aligned}\right\} \tag{4.13}$$

Now $q_\alpha = (\boldsymbol{r}, \boldsymbol{\phi})$, $v_\alpha = (\dot{\boldsymbol{r}}, \dot{\boldsymbol{\phi}})$ where we use the notation $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3) \equiv (\psi, \theta, \phi)$. The expression for the second variation given by (4.11) remains almost unchanged except that now $\delta\boldsymbol{\varphi} = \delta\boldsymbol{\phi}$, $\delta\boldsymbol{w} = \delta\dot{\boldsymbol{r}}$, $\delta\boldsymbol{\sigma} = \delta\dot{\boldsymbol{\phi}} = (\delta\dot{\psi}, \delta\dot{\theta}, \delta\dot{\phi})$, $A_{\alpha\beta} = 0$ and $B_{\alpha\beta}\delta q_\alpha \delta q_\beta = \tilde{B}_{ik}\delta\phi_i\delta\phi_k$ where matrix $[\tilde{B}_{ik}]$ is given by

$$\left[\tilde{B}_{ik}\right] \equiv \begin{pmatrix} 0 & -\boldsymbol{e}_z \cdot (\boldsymbol{x} \times \boldsymbol{n}) & \boldsymbol{e}_y \cdot (\boldsymbol{x} \times \boldsymbol{n}) \\ -\boldsymbol{e}_z \cdot (\boldsymbol{x} \times \boldsymbol{n}) & 0 & -\boldsymbol{e}_x \cdot (\boldsymbol{x} \times \boldsymbol{n}) \\ \boldsymbol{e}_y \cdot (\boldsymbol{x} \times \boldsymbol{n}) & -\boldsymbol{e}_x \cdot (\boldsymbol{x} \times \boldsymbol{n}) & 0 \end{pmatrix}.$$

Moreover, with help of the equilibrium condition (2.11) it can be shown that

$$-\frac{1}{2}\int_{\partial\mathscr{D}_{b0}} P\tilde{B}_{ik}\delta\phi_i\delta\phi_k \mathrm{d}s = \Pi_\psi \delta\theta\delta\phi - \Pi_\theta \delta\psi\delta\phi + \Pi_\phi \delta\psi\delta\theta,$$

where $\Pi_{\phi_i} \equiv \partial\Pi/\partial\phi_i$ at $\boldsymbol{r} = 0$, $\boldsymbol{\phi} = 0$.

### 4.2. $\delta^2 E$ *for a sphere*

Consider a particular case of a spherical body of radius $a$. Clearly, no torque is exerted on the spherical body by an inviscid fluid. We suppose that the potential $\Pi = \Pi(\boldsymbol{r})$ is independent of the Euler angles (i.e. no external moment of force is applied to the body). Then the Euler angles of the body are cyclic coordinates and can therefore be ignored. This means that in (4.11) all terms with the variations of the Euler angles can be discarded and the second variation simplifies to

$$\delta^2 E = \delta^2 E_A + \delta^2 E_c + \delta^2 E_b, \tag{4.14a}$$

$$2\delta^2 E_A = \int_{\mathscr{D}_{f0}} \rho\{(\delta\boldsymbol{u})^2 + \boldsymbol{U}\cdot(\delta\boldsymbol{x}\times\delta\boldsymbol{\omega})\}\mathrm{d}\tau, \tag{4.14b}$$

$$2\delta^2 E_c = \int_{\partial\mathscr{D}_{b0}} \{2\rho\boldsymbol{U}\cdot\delta\boldsymbol{u} - \delta\boldsymbol{r}\cdot\nabla P\}(\delta\boldsymbol{r}\cdot\boldsymbol{n})\mathrm{d}s + \int_{\partial\mathscr{D}_{b0}} \rho(\delta\boldsymbol{r}\cdot\boldsymbol{n})(\delta\boldsymbol{x}\cdot\nabla H)\mathrm{d}s, \tag{4.14c}$$

$$2\delta^2 E_b = M\delta\dot{r}_i\delta\dot{r}_i + \frac{\partial^2 \Pi}{\partial R_i \partial R_k}\delta r_i \delta r_k. \tag{4.14d}$$

If, in addition, the basic flow is such that $\boldsymbol{\Omega}\cdot\boldsymbol{n} = 0$ on $\partial\mathscr{D}_{b0}$, then it can be shown from (2.10) that $H = \text{const}$ on $\partial\mathscr{D}_{b0}$, and (4.14$c$) reduces to the equation

$$2\delta^2 E_c = \int_{\partial\mathscr{D}_{b0}} \rho\{2\boldsymbol{U}\cdot\delta\boldsymbol{u} + \delta\boldsymbol{r}\cdot\nabla G\}(\delta\boldsymbol{r}\cdot\boldsymbol{n})\mathrm{d}s,$$

where $G$ is defined by (4.7).

*Remark A*. All the results described above were obtained for a rigid body placed in an arbitrary rotational inviscid flow. However it is easy to see that these results are equally valid for a rigid body with a cavity containing an ideal fluid. The only difference between these two problems lies in interpreting the boundary $\partial\mathscr{D}_b$, namely, for a body with fluid-filled cavity we consider the surface $\partial\mathscr{D}_b$ as an internal (for the body) boundary which represents the boundary of the cavity, i.e. $\partial\mathscr{D}_b$ is an outer boundary of the fluid domain $\mathscr{D}_f$ which is completely filled with fluid. With this interpretation the basic state given by (2.9)–(2.11) represents an equilibrium of a rigid body with a cavity containing a fluid which in turn is in steady motion with velocity field $\boldsymbol{U}(\boldsymbol{x})$.

*Remark B*. The theory developed in the previous sections can be easily modified to cover the situation when there are $n$ rigid bodies in a fluid or the situation when a cavity in the rigid body contains fluid and other rigid bodies.

## 5. Variational principles for the basic states that are steady relative to a moving frame of reference

The variational principle that we have constructed in § 3 is applicable only to the situation when the whole system 'body + fluid' is confined to some bounded domain $\mathscr{D}$ with rigid boundary $\partial\mathscr{D}$ fixed in space. In the case when the fluid domain extends to infinity the variational principle of § 3 does not work because in the basic state (2.9) the energy of the system is infinite. In general, there are two ways to deal with such a situation. In the first one we could construct some functional (regularized functional) which, first, is an invariant of the equations of motion, second, exists at the basic state and, third, has a critical point at this state. Usually it turns out to be quite a difficult problem. In this section, we shall use another, somewhat more physically motivated, approach to handle the problem. Namely, we shall establish a variational principle for an unsteady state which represents a stationary translational motion of the body along some fixed axis through a fluid that extends to infinity at least in the direction of motion of the body and is at rest there (so that such a state of the system is steady relative to some moving set of axes). Throughout this section we shall consider only a body that moves through surrounding fluid (the case of a body with fluid-filled cavity moving as a whole is trivial).

Let the system 'body + fluid' be invariant with respect to translations along some fixed axis, say the $z$-axis. To guarantee this it is sufficient to assume, first, that the outer boundary $\partial\mathscr{D}$ is invariant under such translations (so that it may be an infinite cylinder of an arbitrary cross-section) and, second, that external forces applied to the system have zero components along the $z$-axis, i.e.

$$\boldsymbol{e}_z \cdot \nabla\Phi = 0, \qquad \partial\Pi\big(\boldsymbol{r},\boldsymbol{\phi}\big)/\partial r_3 = 0. \tag{5.1}$$

Under these assumptions there exists an additional invariant of the problem, a $z$-component of the total momentum of the system

$$N = N_b + N_f = \text{const}, \qquad N_b = M\boldsymbol{e}_z \cdot \dot{\boldsymbol{r}}, \qquad N_f = \int_{\mathscr{D}_f} \rho\, \boldsymbol{e}_z \cdot \boldsymbol{u}\, \mathrm{d}\tau. \tag{5.2}$$

When the fluid extends to infinity in all directions the volume integral in (5.2) that represents the total momentum of the fluid is, in general, not absolutely convergent and depends on the way in which the volume of integration is allowed to tend to infinity. In this case it is natural to use the vortex momentum defined by the equation (Vladimirov 1977)

$$N_\omega = \frac{1}{2}\int_{\mathscr{D}_f} \rho\, \boldsymbol{e}_z \cdot \big(\boldsymbol{x} \times \boldsymbol{\omega}\big)\mathrm{d}\tau - \frac{1}{2}\int_{\partial\mathscr{D}_b} \rho\, \boldsymbol{e}_z \cdot (\boldsymbol{x} \times (\boldsymbol{n} \times \boldsymbol{u}))\mathrm{d}s. \tag{5.3}$$

The volume integral appearing in (5.3) exists and the total momentum of the system 'body + fluid' given by

$$N_1 = N_b + N_\omega \tag{5.4}$$

is conserved provided that (Vladimirov 1977, 1979)

$$r^4|\boldsymbol{\omega}| \to 0, \quad r|\boldsymbol{u}| \to 0 \quad \text{as} \quad r \equiv |\boldsymbol{x}| \to \infty. \tag{5.5a}$$

It can be shown (see Vladimirov 1977, 1979) that in the presence of outer boundaries that are invariant under translations in the $z$-direction (i.e. $\boldsymbol{e}_z \cdot \boldsymbol{n} = 0$ on $\partial\mathscr{D}$) the total momentum (5.4) is also an invariant of the governing equations if $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ on $\partial\mathscr{D}$ (note that this condition is consistent with equations of motion in the sense that if it is satisfied at some initial instant $t = 0$ then it holds for any $t > 0$). For the sake of simplicity we shall assume that the fluid extends to infinity in all directions. The existence of the energy invariant (2.8) then requires that $r^{3/2}|\boldsymbol{u}| \to 0$ as $r \to \infty$ (note that such a behaviour at infinity is certainly satisfied for the most interesting case of a dipole asymptotic). Thus, for the existence of both invariants $N_1$ and $E$ it is sufficient to take the following conditions at infinity†

$$r^4\,\big|\boldsymbol{\omega}\big| \to 0, \quad r^{3/2}\,\big|\boldsymbol{u}\big| \to 0 \quad \text{as} \quad r \to \infty. \tag{5.5b}$$

† In general, it is possible to use condition (5.5*a*) instead of (5.5*b*). In this case we should replace the true energy of the fluid $E_f$ (2.8*b*) by the vortex energy (see e.g. Batchelor 1967). For our problem it is not necessary because the behaviour of the velocity field at infinity due to a moving body is given by a dipole asymptotic $|\boldsymbol{u}| \sim 1/r^3$.

### 5.1. *Basic state*

The basic state whose stability will be studied is given by

$$\left.\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{U}\big(\boldsymbol{x}-\boldsymbol{R}(t)\big), \quad \boldsymbol{w}=\boldsymbol{W}(t)=U_0\boldsymbol{e}_z, \quad \boldsymbol{\sigma}=\boldsymbol{\Sigma}=0,\\ \boldsymbol{r}(t) &= \boldsymbol{R}(t)=U_0 t\boldsymbol{e}_z, \quad \psi=0, \quad \theta=0, \quad \phi=0. \end{aligned}\right\} \tag{5.6}$$

Here $\psi$, $\theta$ and $\phi$ are the Euler angles introduced in § 4; $U_0$ is a constant velocity of a translational motion of the body (along the $z$-axis). The velocity field $\boldsymbol{U}$ is a solution of the problem

$$\left.\begin{aligned} &\boldsymbol{\Omega}\times\boldsymbol{U}-U_0\big(\boldsymbol{e}_z\cdot\nabla\big)\boldsymbol{U}=-\nabla H, \quad \nabla\cdot\boldsymbol{U}=0 \quad \text{in } \mathscr{D}_{f0}(t);\\ &\boldsymbol{U}\cdot\boldsymbol{n}=U_0\boldsymbol{e}_z\cdot\boldsymbol{n} \quad \text{on } \partial\mathscr{D}_{f0}(t), \quad r^{3/2}\,|\boldsymbol{U}|\to 0 \text{ and } r^4\,|\boldsymbol{\Omega}|\to 0 \quad \text{as } r\to\infty, \end{aligned}\right\} \tag{5.7}$$

where $H \equiv P/\rho + \Phi + \frac{1}{2}\boldsymbol{U}^2$. Note that as indicated in (5.7) the fluid domain in the basic state now depends on time.

In the basic state, the force and torque exerted on the body by the fluid are balanced by the externally applied force and torque:

$$-\partial\Pi/\partial\boldsymbol{r} + \int_{\partial\mathscr{D}_{b0}} P\boldsymbol{n}\,\mathrm{d}s = 0, \tag{5.8a}$$

$$-\partial\Pi/\partial\boldsymbol{\phi} + \int_{\partial\mathscr{D}_{b0}} P((\boldsymbol{x}-\boldsymbol{R})\times\boldsymbol{n})\,\mathrm{d}s = 0. \tag{5.8b}$$

Here, in accordance with our assumption (5.1), $\Pi = \Pi\big(r_1, r_2, \boldsymbol{\phi}\big)$; the derivatives $\partial\Pi/\partial\boldsymbol{r}$ and $\partial\Pi/\partial\boldsymbol{\phi}$ are evaluated at the point $r_1 = R_1$, $r_2 = R_2$, $\boldsymbol{\phi} = 0$.

### 5.2. *Variational principle*

As in § 3 we introduce the family of transformations

$$\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}}(\boldsymbol{x}(\boldsymbol{a},t),\epsilon), \quad \tilde{\boldsymbol{r}} = \tilde{\boldsymbol{r}}(t,\epsilon), \quad \tilde{\boldsymbol{\phi}} = \tilde{\boldsymbol{\phi}}(t,\epsilon) \tag{5.9}$$

depending on a parameter $\epsilon \geqslant 0$ where functions $\tilde{\boldsymbol{x}}(\boldsymbol{x},\epsilon)$, $\tilde{\boldsymbol{r}}(t,\epsilon)$ and $\tilde{\boldsymbol{\phi}}(t,\epsilon)$ are twice differentiable with respect to $\epsilon$ and the value $\epsilon = 0$ corresponds to the exact solution (5.6):

$$\tilde{\boldsymbol{x}}(\boldsymbol{x}(\boldsymbol{a},t),0) = \boldsymbol{x}(\boldsymbol{a},t), \quad \tilde{\boldsymbol{r}}(t,0) = \boldsymbol{R}(t), \quad \tilde{\boldsymbol{\phi}}(t,0) = 0. \tag{5.10}$$

In (5.9), (5.10) $\boldsymbol{x}(\boldsymbol{a},t) \in \partial\mathscr{D}_{f0}(t)$ and $\boldsymbol{a}$ is the Lagrangian coordinate (label) of a fluid element. For any fixed moment of time $t$ the transformations defined by (5.9)–(5.10) can be interpreted as a 'virtual motion' of the system 'body + fluid' with a 'virtual time' $\epsilon$; $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathscr{D}_{f0}$) serves as a label to identify the fluid particle, while $\tilde{\boldsymbol{x}}(\boldsymbol{x},\epsilon)$ represents its trajectory. The domain $\mathscr{D}_{f0}(t) = \tilde{\mathscr{D}}_f(t,0)$ evolves to a new one $\tilde{\mathscr{D}}_f(t,\epsilon)$ which is completely determined by the position and the orientation of the rigid body, i.e. by $\tilde{\boldsymbol{r}}(t,\epsilon)$ and $\tilde{\boldsymbol{\phi}}(t,\epsilon)$. $\tilde{\boldsymbol{x}}(\boldsymbol{x},\epsilon)$, $\tilde{\boldsymbol{r}}(t,\epsilon)$ and $\tilde{\boldsymbol{\phi}}(t,\epsilon)$ are specified through functions $\boldsymbol{f}(\tilde{\boldsymbol{x}},t,\epsilon)$, $\boldsymbol{h}(t,\epsilon)$ and $\boldsymbol{g}(t,\epsilon)$ by the same equations as (3.3):

$$\tilde{\boldsymbol{x}}_\epsilon = \boldsymbol{f}(\tilde{\boldsymbol{x}},t,\epsilon), \quad \tilde{\boldsymbol{r}}_\epsilon = \boldsymbol{h}(t,\epsilon), \quad \tilde{\boldsymbol{\phi}}_\epsilon = \boldsymbol{g}(t,\epsilon), \tag{5.11}$$

where $\boldsymbol{h}(t,\epsilon)$, $\boldsymbol{g}(t,\epsilon)$ are arbitrary differentiable (with respect to $\epsilon$) functions and $\boldsymbol{f}(\tilde{\boldsymbol{x}},t,\epsilon)$ is an arbitrary vector field differentiable with respect to $\epsilon$ and satisfying the conditions†

$$\tilde{\nabla} \cdot \boldsymbol{f} = 0 \quad \text{in} \quad \tilde{\mathscr{D}}_f(t, \epsilon), \tag{5.12a}$$

$$\boldsymbol{f} \cdot \boldsymbol{n} = \left[\tilde{\boldsymbol{r}}_\epsilon + \tilde{\boldsymbol{\varphi}}_\epsilon \times \left(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}}\right)\right] \cdot \boldsymbol{n} \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(t, \epsilon), \tag{5.12b}$$

$$\left|\boldsymbol{f}\right| \to 0 \quad \text{as} \quad \tilde{r} \to \infty. \tag{5.12c}$$

In (5.12) we used the notation

$$\tilde{\varphi}_{i\epsilon} \equiv -\tfrac{1}{2} e_{ijk} \frac{\partial \tilde{P}_{jl}}{\partial \tilde{\phi}_m} \tilde{P}_{kl} \phi_{m\epsilon} = -\tfrac{1}{2} e_{ijk} \frac{\partial \tilde{P}_{jl}}{\partial \tilde{\phi}_m} \tilde{P}_{kl} g_m. \tag{5.13}$$

The actual velocity field of the fluid and the actual generalized velocity of the rigid body in 'virtual motion' are described by twice differentiable (with respect to $\epsilon$) functions $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, t, \epsilon)$, $\tilde{\boldsymbol{w}}(t, \epsilon)$ and $\tilde{\boldsymbol{\zeta}}(t, \epsilon)$ such that the value $\epsilon = 0$ corresponds to the steady state (5.6):

$$\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, t, \epsilon)|_{\epsilon=0} = \boldsymbol{U}(\tilde{\boldsymbol{x}}, t), \quad \tilde{\boldsymbol{w}}(t, \epsilon)|_{\epsilon=0} = \boldsymbol{W}(t), \quad \tilde{\boldsymbol{\zeta}}(t, \epsilon)|_{\epsilon=0} = \boldsymbol{\zeta}(t) \equiv d\boldsymbol{\phi}/dt = 0. \tag{5.14}$$

In addition, the field $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, t, \epsilon)$ satisfies the conditions

$$\tilde{\nabla} \cdot \tilde{\boldsymbol{u}} = 0 \quad \text{in} \quad \tilde{\mathscr{D}}_f(t, \epsilon), \tag{5.15a}$$

$$\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = (\tilde{\boldsymbol{w}} + \tilde{\boldsymbol{\sigma}} \times (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}})) \cdot \boldsymbol{n} \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(t, \epsilon), \tag{5.15b}$$

$$r^{3/2} \left|\tilde{\boldsymbol{u}}\right| \to 0, \;\; r^4 \left|\tilde{\boldsymbol{\omega}}\right| \to 0 \quad \text{as} \quad r \to \infty. \tag{5.15c}$$

Here $\tilde{\boldsymbol{\sigma}}$ as a function of $\tilde{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{\zeta}}$ is given by (4.13). The evolution with the 'time' $\epsilon$ of the generalized velocities $\tilde{\boldsymbol{w}}(t, \epsilon)$ and $\tilde{\boldsymbol{\zeta}}(t, \epsilon)$ is prescribed by the equations

$$\tilde{\boldsymbol{w}}_\epsilon = \boldsymbol{h}^*(t, \epsilon), \quad \tilde{\boldsymbol{\zeta}}_\epsilon = \boldsymbol{g}^*(t, \epsilon) \tag{5.16}$$

with some differentiable functions $\boldsymbol{g}^*(t, \epsilon)$ and $\boldsymbol{h}^*(t, \epsilon)$. Note that the functions $\boldsymbol{g}(t, \epsilon)$, $\boldsymbol{h}(t, \epsilon)$, $\boldsymbol{g}^*(t, \epsilon)$ and $\boldsymbol{h}^*(t, \epsilon)$ which determine the evolution in the 'virtual motion' of the generalized velocities and coordinates are all arbitrary, so that $\tilde{\boldsymbol{\phi}}(t, \epsilon)$, $\tilde{\boldsymbol{r}}(t, \epsilon)$, $\tilde{\boldsymbol{\zeta}}(t, \epsilon)$ and $\tilde{\boldsymbol{w}}(t, \epsilon)$ vary independently.

The evolution of the field $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, t, \epsilon)$ is defined by the equation

$$\tilde{\boldsymbol{u}}_\epsilon = \boldsymbol{f} \times \tilde{\boldsymbol{\omega}} + \tilde{\nabla}\alpha \tag{5.17}$$

with some function $\alpha(\tilde{\boldsymbol{x}}, t, \epsilon)$ which, as follows from (5.15), (5.17), can be found by solving the problem (cf. (3.11))

$$\tilde{\nabla}^2\alpha = -\tilde{\nabla} \cdot \left(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}\right) \quad \text{in} \quad \tilde{\mathscr{D}}_f(t, \epsilon), \tag{5.18a}$$

$$\boldsymbol{n} \cdot \tilde{\nabla}\alpha = \tilde{\boldsymbol{u}}_\epsilon \cdot \boldsymbol{n} - \boldsymbol{n} \cdot \left(\boldsymbol{f} \times \tilde{\boldsymbol{\omega}}\right) \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(t, \epsilon), \tag{5.18b}$$

$$\tilde{r}^{3/2} \left|\tilde{\nabla}\alpha\right| \to 0 \quad \text{as} \quad \tilde{r} \to \infty. \tag{5.18c}$$

Consider now the conserved functional

$$\mathscr{I} = E + \lambda N_1, \tag{5.19}$$

with an arbitrary constant $\lambda$. We shall show that with a certain choice of $\lambda$ the first variation $\delta\mathscr{I}$ evaluated at the basic state (5.6) vanishes, i.e. this functional has a critical point.

† In principle, the condition (5.12$c$) can be replaced by a weaker one: $|\boldsymbol{f}| \to \text{const}$ as $\tilde{r} \to \infty$.

The calculation of the first variation of $E$ is similar to that of § 3 and results in

$$\delta E = \delta E_f + \delta E_b, \quad \delta E_b \equiv M\boldsymbol{W}\cdot\delta\boldsymbol{w} + \delta\Pi, \tag{5.20a}$$

$$\delta\Pi \equiv \frac{\partial\Pi}{\partial R_1}\delta R_1 + \frac{\partial\Pi}{\partial R_2}\delta R_2 + \frac{\partial\Pi}{\partial\boldsymbol{\phi}}\delta\boldsymbol{\phi}, \tag{5.20b}$$

$$\delta E_f \equiv \int_{\mathscr{D}_{f0}} \rho U_0\{\delta\boldsymbol{x}\cdot\nabla(\boldsymbol{e}_z\cdot\boldsymbol{U}) + \delta\boldsymbol{x}\cdot[\boldsymbol{\Omega}\times\boldsymbol{e}_z]\}\mathrm{d}\tau - \int_{\partial\mathscr{D}_{b0}}\{P(\delta\boldsymbol{y}\cdot\boldsymbol{n}) + \rho U_0\alpha(\boldsymbol{e}_z\cdot\boldsymbol{n})\}\mathrm{d}s. \tag{5.20c}$$

The first variation of the momentum $N$ is given by

$$\left.\begin{aligned} \delta N &= \delta N_f + \delta N_b, \quad \delta N_b \equiv M\boldsymbol{e}_z\cdot\delta\boldsymbol{w}, \\ \delta N_f &\equiv \textstyle\int_{\mathscr{D}_{f0}} \rho U_0\delta\boldsymbol{x}\cdot[\boldsymbol{\Omega}\times\boldsymbol{e}_z]\mathrm{d}\tau + \int_{\partial\mathscr{D}_{b0}}\rho\{(\delta\boldsymbol{y}\cdot\boldsymbol{n})(\boldsymbol{e}_z\cdot\boldsymbol{U}) - \alpha(\boldsymbol{e}_z\cdot\boldsymbol{n})\}\mathrm{d}s. \end{aligned}\right\} \tag{5.21}$$

In the derivation of (5.21) we have used the well known rule of differentiating an integral over a material surface. Now it follows from (5.8), (5.20), (5.21) that

$$\delta\mathscr{I} = 0 \quad \text{if} \quad \lambda = -U_0.$$

Thus, we have shown that the functional $\mathscr{I}$ with $\lambda = -U_0$ has a critical point at any solution of the form (5.6) provided that we take account only of 'isovortical' fluid flows.

### 5.3. *The second variation*

The procedure for calculating $\delta^2\mathscr{I}$ is analogous to our calculations in § 4. The only new feature is that we need to calculate the second derivative with respect to $\epsilon$ of a material surface integral. It can be shown by applying twice the usual formula for the rate of change of an integral over a material surface (see e.g. Batchelor 1967) that for any sufficiently smooth field $\boldsymbol{F}(\tilde{\boldsymbol{x}},\epsilon)$ the following equality is valid:

$$\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\int_{\partial\tilde{\mathscr{D}}_b(\epsilon)} \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\partial\tilde{\mathscr{D}}_b(\epsilon)}\{\boldsymbol{n}\cdot\boldsymbol{F}_{\epsilon\epsilon} + 2(\boldsymbol{n}\cdot\boldsymbol{f})\tilde{\nabla}\cdot\boldsymbol{F}_\epsilon + (\boldsymbol{n}\cdot\boldsymbol{f}_\epsilon)\tilde{\nabla}\cdot\boldsymbol{F} + (\boldsymbol{n}\cdot\boldsymbol{f})(\boldsymbol{f}\cdot\tilde{\nabla})\tilde{\nabla}\cdot\boldsymbol{F}\}\mathrm{d}s.$$

By using this formula and following the same procedure as in § 4 we arrive at the expression

$$\delta^2\mathscr{I} = \delta^2\mathscr{I}_A + \delta^2\mathscr{I}_c + \delta^2\mathscr{I}_b, \tag{5.22a}$$

$$2\delta^2\mathscr{I}_A \equiv \int_{\mathscr{D}_{f0}} \rho[(\delta\boldsymbol{u})^2 + \boldsymbol{U}^*\cdot(\delta\boldsymbol{x}\times\delta\boldsymbol{\omega})]\mathrm{d}\tau, \tag{5.22b}$$

$$2\delta^2\mathscr{I}_c \equiv \int_{\partial\mathscr{D}_{f0}} [2\rho\boldsymbol{U}^*\cdot\boldsymbol{u} - \delta\boldsymbol{y}\cdot\nabla P](\delta\boldsymbol{y}\cdot\boldsymbol{n})\mathrm{d}s + \int_{\partial\mathscr{D}_{f0}}\rho[\delta\boldsymbol{x}\cdot\nabla H^*](\delta\boldsymbol{y}\cdot\boldsymbol{n})\mathrm{d}s - \int_{\partial\mathscr{D}_{f0}} P\boldsymbol{n}\cdot(\delta\boldsymbol{r}\times\delta\boldsymbol{\phi})\mathrm{d}s, \tag{5.22c}$$

$$2\delta^2\mathscr{I}_b \equiv M\big(\delta \boldsymbol{w}\big)^2 + \delta\boldsymbol{\sigma}\cdot\hat{\boldsymbol{I}}\cdot\delta\boldsymbol{\sigma} + \frac{\partial^2 \Pi}{\partial R_i \partial R_k}\delta R_i \delta R_k + \frac{\partial^2 \Pi}{\partial \phi_i \partial \phi_k}\delta\phi_i\delta\phi_k$$

$$+2\frac{\partial^2 \Pi}{\partial R_i \partial \phi_k}\delta r_i \delta\phi_k + 2\{\Pi_\psi \delta\theta\delta\phi - \Pi_\theta \delta\psi\delta\phi + \Pi_\phi \delta\psi\delta\theta\}, \qquad (5.22d)$$

where $\boldsymbol{U}^* \equiv \boldsymbol{U} - U_0\boldsymbol{e}_z$, $H^* \equiv P/\rho + \Phi + \frac{1}{2}|\boldsymbol{U}^*|^2$ and where

$$\Pi_\psi \equiv \partial\Pi/\partial\psi\big|_{\phi=0}, \quad \Pi_\theta \equiv \partial\Pi/\partial\theta\big|_{\phi=0}, \quad \Pi_\phi \equiv \partial\Pi/\partial\phi\big|_{\phi=0}.$$

It can be shown that in the presence of fixed rigid boundaries that are invariant with respect to translations along the $z$-axis the second variation of $\mathscr{I}$ is also given by (5.22). We conclude then that the stability problem for a body moving through a fluid that extends to infinity in some or all directions is reduced to the search for the conditions under which the second variation (5.22) is of definite sign.

## 6. Variational principles for the basic states that are steady relative to a steadily rotating reference frame

It is known that if a dynamical system is invariant with respect to rotations round an axis then it has an additional constant of motion, the projection on this axis of the angular momentum of the system. In this case it is possible to establish a variational principle for such a state of the system that is steady relative to a frame of reference rotating round the axis with constant angular velocity. Such a principle will be established in this section.

Consider the situation when no external moment of force along a fixed axis, say the $z$-axis, is applied to the system 'body + fluid'. In other words, we suppose that

$$\boldsymbol{e}_z \cdot \big[\boldsymbol{x} \times \nabla\Phi\big] = 0, \qquad (6.1)$$

and that $\Pi$ does not depend on the Euler angle $\phi$: $\partial\Pi/\partial\phi = 0$ (i.e. $\phi$ is a cyclic coordinate). Throughout this section the system 'body + fluid' will be interpreted, in accordance with Remark A of §4, as a rigid body with a cavity containing a fluid. Under these assumptions there exists an additional invariant of the problem, a $z$-component of the angular momentum of the system:†

$$L = L_b + L_f = \boldsymbol{e}_z \cdot (M\boldsymbol{r} \times \dot{\boldsymbol{r}} + \hat{\boldsymbol{I}}\cdot\boldsymbol{\sigma}) + \int_{\mathscr{D}_f} \rho\, \boldsymbol{e}_z \cdot \big[\boldsymbol{x} \times \boldsymbol{u}\big]\mathrm{d}\tau = \text{const}. \qquad (6.2)$$

### 6.1. *Basic state*

As a basic state whose stability will be studied we consider the state of the system that is steady relative to the coordinate system rotating with a constant angular velocity $\sigma_0$. In axes fixed in space the basic state is given by

$$\left.\begin{aligned} &\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x},t), \quad \boldsymbol{w} = \boldsymbol{W}(t), \quad \boldsymbol{\sigma} = \boldsymbol{\Sigma} = \sigma_0\boldsymbol{e}_z, \\ &\boldsymbol{r} = \boldsymbol{R}(t), \quad \psi = 0, \quad \theta = 0, \quad \phi = \sigma_0 t. \end{aligned}\right\} \qquad (6.3)$$

Since relative to the rotating set of axes $Ox_1'x_2'x_3'$ the basic state is steady we have

$$\partial\boldsymbol{U}'(\boldsymbol{x}',t)/\partial t = 0, \qquad (6.4)$$

† For conservation of the angular momentum in the case of a body placed in a fluid, the (fixed) outer boundary $\partial\mathscr{D}$ of the fluid domain must be axisymmetric.

where, by definition,

$$\boldsymbol{U} \equiv \hat{\boldsymbol{P}} \cdot \boldsymbol{U}' \quad \text{and} \quad \boldsymbol{x} \equiv \hat{\boldsymbol{P}} \cdot \boldsymbol{x}' \tag{6.5a}$$

with rotation matrix $\hat{\boldsymbol{P}}$ given by

$$\hat{\boldsymbol{P}} = \begin{pmatrix} \cos(\sigma_0 t) & \sin(\sigma_0 t) & 0 \\ -\sin(\sigma_0 t) & \cos(\sigma_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.5b}$$

In (6.4), (6.5), $U_i'$ $(i = 1, 2, 3)$ represent the components of the absolute velocity vector $\boldsymbol{U}$ relative to the rotating set of axes $Ox_1'x_2'x_3'$. On differentiating (6.5*a*) with respect to time, we obtain

$$\partial \boldsymbol{U}(\boldsymbol{x}, t)/\partial t = \sigma_0[\boldsymbol{e}_z \times \boldsymbol{U}] - \sigma_0[(\boldsymbol{e}_z \times \boldsymbol{x}) \cdot \nabla]\boldsymbol{U}. \tag{6.6}$$

After substitution this in (2.1) we find that in the basic state the velocity field $\boldsymbol{U}$ is a solution of the problem

$$\left.\begin{aligned} \boldsymbol{\Omega} \times \boldsymbol{U} + \sigma_0[\boldsymbol{e}_z \times \boldsymbol{U}] - \sigma_0[(\boldsymbol{e}_z \times \boldsymbol{x}) \cdot \nabla]\boldsymbol{U} = -\nabla H, \quad \nabla \cdot \boldsymbol{U} = 0 \quad \text{in } \mathscr{D}_{f0}(t); \\ \boldsymbol{U} \cdot \boldsymbol{n} = \sigma_0[\boldsymbol{e}_z \times \boldsymbol{x}] \cdot \boldsymbol{n} \quad \text{on } \partial\mathscr{D}_{f0}(t), \end{aligned}\right\} \tag{6.7}$$

where $H \equiv P + \Phi + \frac{1}{2}\boldsymbol{U}^2$. Here as in the previous section the fluid domain in the basic state depends on time.

In the basic state, the force and torque exerted on the body by the fluid are balanced by the externally applied force and torque:

$$M\sigma_0^2(\boldsymbol{e}_z \times (\boldsymbol{e}_z \times \boldsymbol{R})) = -\frac{\partial \Pi}{\partial \boldsymbol{r}} + \int_{\partial\mathscr{D}_{b0}} P\boldsymbol{n}\,\mathrm{d}s, \tag{6.8a}$$

$$\int_{\partial\mathscr{D}_{b0}} P\boldsymbol{e}_z \cdot ((\boldsymbol{x} - \boldsymbol{R}) \times \boldsymbol{n})\mathrm{d}s = 0, \tag{6.8b}$$

$$\frac{\partial \boldsymbol{\Sigma}}{\partial \dot{\psi}} \cdot (\boldsymbol{\Sigma} \times (\hat{\boldsymbol{I}} \cdot \boldsymbol{\Sigma})) = -\frac{\partial \Pi}{\partial \psi} + \int_{\partial\mathscr{D}_{b0}} P\frac{\partial \boldsymbol{\Sigma}}{\partial \dot{\psi}} \cdot ((\boldsymbol{x} - \boldsymbol{R}) \times \boldsymbol{n})\mathrm{d}s, \tag{6.8c}$$

$$\frac{\partial \boldsymbol{\Sigma}}{\partial \dot{\theta}} \cdot (\boldsymbol{\Sigma} \times (\hat{\boldsymbol{I}} \cdot \boldsymbol{\Sigma})) = -\frac{\partial \Pi}{\partial \theta} + \int_{\partial\mathscr{D}_{b0}} P\frac{\partial \boldsymbol{\Sigma}}{\partial \dot{\theta}} \cdot ((\boldsymbol{x} - \boldsymbol{R}) \times \boldsymbol{n})\mathrm{d}s, \tag{6.8d}$$

where the derivatives with respect to $\boldsymbol{r}$, $\boldsymbol{\phi}$ and $\dot{\boldsymbol{\phi}}$ are taken at the point $\boldsymbol{r} = \boldsymbol{R}$, $\boldsymbol{\phi} = (0, 0, \sigma_0 t)$, in particular:

$$\partial \boldsymbol{\Sigma}/\partial \dot{\psi} = (\cos(\sigma_0 t), \sin(\sigma_0 t), 0), \quad \partial \boldsymbol{\Sigma}/\partial \dot{\theta} = (-\sin(\sigma_0 t), \cos(\sigma_0 t), 0). \tag{6.8e}$$

### 6.2. *Variational principle*

Consider now the conserved functional

$$\mathscr{I} = E + \lambda L, \tag{6.9}$$

with some constant $\lambda$. We shall show that with a certain choice of $\lambda$ the first variation $\delta^1\mathscr{I}$ evaluated at the basic state (6.3) vanishes.

Calculation of the first variation of $E$ is similar to that of § 5 and results in

$$\left.\begin{aligned} &\delta E = \delta E_f + \delta E_b, \quad \delta E_b \equiv M\boldsymbol{W} \cdot \delta\boldsymbol{w} + I_{ik}\Sigma_k\delta\sigma_i - \delta\boldsymbol{\varphi} \cdot [\boldsymbol{\Sigma} \times (\hat{\boldsymbol{I}} \cdot \boldsymbol{\Sigma})], \\ &\delta E_f \equiv \textstyle\int_{\mathscr{D}_{f0}} \rho\sigma_0\delta\boldsymbol{x} \cdot \{[(\boldsymbol{e}_z \times \boldsymbol{x})]\boldsymbol{U} - \boldsymbol{e}_z \times \boldsymbol{U}\}\mathrm{d}\tau + \int_{\partial\mathscr{D}_{b0}}\{\rho\alpha\boldsymbol{U} \cdot \boldsymbol{n} - P\delta\boldsymbol{x} \cdot \boldsymbol{n}\}\mathrm{d}s. \end{aligned}\right\} \tag{6.10}$$

The first variation of the angular momentum $L$ is given by the equation

$$\delta L = \delta L_f + \delta L_b, \tag{6.11a}$$

$$\delta L_b \equiv M\boldsymbol{e}_z \cdot [\delta \boldsymbol{r} \times \boldsymbol{W}] + M\boldsymbol{e}_z \cdot [\boldsymbol{R} \times \delta \boldsymbol{w}] + \boldsymbol{e}_z \cdot \hat{\boldsymbol{I}} \cdot \delta\boldsymbol{\sigma} + [\boldsymbol{\Sigma} \times \delta\boldsymbol{\varphi}] \cdot \hat{\boldsymbol{I}} \cdot \boldsymbol{e}_z - [\delta\boldsymbol{\varphi} \times \boldsymbol{e}_z] \cdot \hat{\boldsymbol{I}} \cdot \boldsymbol{\Sigma}, \tag{6.11b}$$

$$\delta L_f \equiv \int_{\mathscr{D}_{f0}} \rho\, \delta\boldsymbol{x} \cdot \{[(\boldsymbol{e}_z \times \boldsymbol{x})]\boldsymbol{U} - \boldsymbol{e}_z \times \boldsymbol{U}\}\mathrm{d}\tau + \int_{\partial\mathscr{D}_{b0}} \rho\alpha\, \boldsymbol{n} \cdot [\boldsymbol{e}_z \times \boldsymbol{x}]\mathrm{d}s. \tag{6.11c}$$

Using (6.8), we find from (6.10), (6.11) that

$$\delta\mathscr{I} = 0 \quad \text{if} \quad \lambda = -\sigma_0.$$

We have thus shown that on the set of 'isovortical' flows the functional $\mathscr{I}$ with $\lambda = -\sigma_0$ has a critical point at any solution of the form (6.3).

### 6.3. *The second variation*

Similar to those in §§4 and 5 the procedure for calculating the second variation of $\mathscr{I}$ can be shown to give the following expression:

$$\delta^2\mathscr{I} = \delta^2\mathscr{I}_A + \delta^2\mathscr{I}_c + \delta^2\mathscr{I}_b, \tag{6.12a}$$

$$2\delta^2\mathscr{I}_A \equiv \int_{\mathscr{D}_{f0}} \rho[(\delta\boldsymbol{u})^2 + \boldsymbol{U}^* \cdot (\delta\boldsymbol{x} \times \delta\boldsymbol{\omega})]\mathrm{d}\tau, \tag{6.12b}$$

$$2\delta^2\mathscr{I}_c \equiv \int_{\partial\mathscr{D}_{b0}} [2\rho\boldsymbol{U}^* \cdot \boldsymbol{u} - \delta\boldsymbol{y} \cdot \nabla P](\delta\boldsymbol{y} \cdot \boldsymbol{n})\mathrm{d}s + \int_{\partial\mathscr{D}_{b0}} \rho[\delta\boldsymbol{y} \cdot \nabla(H - \sigma_0 h)](\delta\boldsymbol{y} \cdot \boldsymbol{n})\mathrm{d}s - \int_{\partial\mathscr{D}_{b0}} P\boldsymbol{n} \cdot (\delta\boldsymbol{r} \times \delta\boldsymbol{\varphi})\mathrm{d}s, \tag{6.12c}$$

$$2\delta^2\mathscr{I}_b \equiv M(\delta\boldsymbol{w})^2 + \delta\boldsymbol{\sigma} \cdot \hat{\boldsymbol{I}} \cdot \delta\boldsymbol{\sigma} + \frac{\partial^2 \Pi}{\partial Q_\alpha \partial Q_\beta}\delta q_\alpha \delta q_\beta - 2M\sigma_0\boldsymbol{e}_z \cdot [\delta\boldsymbol{r} \times \delta\boldsymbol{w}] - \sigma_0^2\{(\boldsymbol{e}_z \cdot \delta\boldsymbol{\varphi})(\delta\boldsymbol{\varphi} \cdot \hat{\boldsymbol{I}} \cdot \boldsymbol{e}_z) - (\delta\boldsymbol{\varphi})^2(\boldsymbol{e}_z \cdot \hat{\boldsymbol{I}} \cdot \boldsymbol{e}_z) + (\boldsymbol{e}_z \times \delta\boldsymbol{\varphi}) \cdot \hat{\boldsymbol{I}} \cdot (\boldsymbol{e}_z \times \delta\boldsymbol{\varphi})\} + 2\delta\phi(\delta\theta\Pi_\psi - \delta\psi\Pi_\theta), \tag{6.12d}$$

where

$$\boldsymbol{U}^* \equiv \boldsymbol{U} - \sigma_0(\boldsymbol{e}_z \times \boldsymbol{x}), \quad h \equiv \boldsymbol{U} \cdot (\boldsymbol{e}_z \times \boldsymbol{x}).$$

*Remark*. Evidently, all results of this section are valid in the case of a body placed in an axisymmetric domain $\mathscr{D}$ filled with a fluid, the corresponding second variation being given by (6.12).

## 7. Variational principle for a two-dimensional problem

In this section we consider the simpler situation when the body is an infinite cylinder with an arbitrary cross-section moving perpendicularly to its axis and the flow is two-dimensional, i.e. it does not depend on the coordinate along the axis of a cylinder. In two-dimensional motion the position and the orientation of the body can be described by three independent quantities: two Cartesian components of the radius-vector of the centre of mass of the body $\boldsymbol{r} = (r_1, r_2)$ and an angle $\phi$ that represents a rotation of the body around the $z$-axis (which is perpendicular to the plane of motion). The equations of motion of the body reduce to

$$M\dot{w}_i \equiv M\ddot{r}_i = \int_{\partial\mathscr{D}_b} pn_i \mathrm{d}l - \partial\Pi(\boldsymbol{r},\phi)/\partial r_i, \tag{7.1a}$$

$$I\dot{\sigma} \equiv I\ddot{\phi} = \int_{\partial\mathscr{D}_b} \boldsymbol{e}_z \cdot [(\boldsymbol{x}-\boldsymbol{r})\times\boldsymbol{n}]p\mathrm{d}l - \partial\Pi(\boldsymbol{r},\phi)/\partial\phi, \tag{7.1b}$$

where $I$ is the moment of inertia of the body ($I$ represents an $I_{zz}$-component of the inertia tensor $I_{ik}$). The equations of motion of the fluid (2.1) remain the same except that the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ now has only two non-zero components $\boldsymbol{u}=(u_1,u_2)$ depending on two Cartesian coordinates $\boldsymbol{x}=(x_1,x_2)$; the boundary conditions (2.7) take the form

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\mathscr{D}, \tag{7.2a}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = (\boldsymbol{w}+\sigma\boldsymbol{e}_z\times(\boldsymbol{x}-\boldsymbol{r}))\cdot\boldsymbol{n} \quad \text{on} \quad \partial\mathscr{D}_b. \tag{7.2b}$$

Note that equation (2.2) for vorticity in the two-dimensional case simplifies to

$$\mathrm{D}\omega = 0$$

where $\omega \equiv \boldsymbol{e}_z\cdot\nabla\times\boldsymbol{u}$ is a $z$-component of the vorticity.

The global invariants of (2.1), (7.1) with boundary conditions (7.2) are the total energy

$$E = E_f + E_b = \text{const}, \tag{7.3a}$$

$$E_f \equiv \int_{\mathscr{D}_f} \rho\left(\tfrac{1}{2}\boldsymbol{u}^2 + \Phi\right)\mathrm{d}\tau, \tag{7.3b}$$

$$E_b \equiv \tfrac{1}{2}M\boldsymbol{w}^2 + \tfrac{1}{2}I\sigma^2 + \Pi(\boldsymbol{r},\phi), \tag{7.3c}$$

the Casimir invariant

$$\mathscr{C} = \int_{\tau-\tau_b} \rho F(\omega)\mathrm{d}\tau = \text{const}, \tag{7.4}$$

and circulations of velocity round the closed curves $\partial\mathscr{D}$ and $\partial\mathscr{D}_b$

$$\Gamma_0 = \int_{\partial\mathscr{D}} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l} = \text{const}, \quad \Gamma_b = \int_{\partial\mathscr{D}_b} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l} = \text{const}. \tag{7.5}$$

$F(\omega)$ that appears in (7.4) is an arbitrary smooth function.

### 7.1. *Basic state*

An exact steady solution of (2.1), (7.1), (7.2) whose stability will be investigated is a two-dimensional analogue of (2.9) given by

$$\boldsymbol{r}=0, \quad \phi=0, \quad \boldsymbol{w}=0, \quad \sigma=0; \tag{7.6a}$$

$$\boldsymbol{u}=\boldsymbol{U}(\boldsymbol{x}) \quad \text{in} \quad \mathscr{D}_{f0}. \tag{7.6b}$$

The velocity field $\boldsymbol{U}(\boldsymbol{x})$ in (7.6) is a solution of the problem

$$(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = -\frac{1}{\rho}\nabla P - \nabla\Phi, \quad \nabla\cdot\boldsymbol{U}=0 \quad \text{in} \quad \mathscr{D}_{f0}; \tag{7.7a}$$

$$\boldsymbol{U}\cdot\boldsymbol{n}=0 \quad \text{on} \quad \partial\mathscr{D}_{b0} \quad \text{and} \quad \partial\mathscr{D}. \tag{7.7b}$$

At equilibrium, the total force and torque exerted by the fluid on the body are balanced by the external force and torque applied to the body (cf. (2.11)):

$$\partial\Pi/\partial \boldsymbol{r}|_{r,\phi=0} = \int_{\partial\mathscr{D}_{b0}} P\boldsymbol{n}\,\mathrm{d}l, \quad \partial\Pi/\partial\phi|_{r,\phi=0} = \int_{\mathscr{D}_{b0}} P\boldsymbol{e}_z\cdot(\boldsymbol{x}\times\boldsymbol{n})\mathrm{d}l. \tag{7.8}$$

As usual, in the two-dimensional case we introduce the stream function $\Psi(\boldsymbol{x})$ of the basic flow defined by the equation

$$\boldsymbol{U} \equiv -\nabla\times(\Psi\boldsymbol{e}_z), \tag{7.9}$$

so that the vorticity of the basic flow is given by the equation $\Omega \equiv \boldsymbol{e}_z\cdot\nabla\times\boldsymbol{U} = \Delta\Psi$. The *curl* of (7.7*a*) simplifies to the equation

$$\boldsymbol{U}\cdot\nabla\Omega = 0.$$

Therefore,

$$\Omega = \Omega(\Psi). \tag{7.10$a$}$$

Equation (7.7*a*) can be rewritten in the form

$$\Omega\nabla\Psi = \nabla H, \quad H \equiv G + P/\rho, \quad G \equiv \Phi + U_iU_i/2, \tag{7.10$b$}$$

from which it follows that

$$H = H(\Psi), \quad \Omega(\Psi) = \mathrm{d}H/\mathrm{d}\Psi. \tag{7.10$c$}$$

If $\Omega' \equiv \mathrm{d}\Omega/\mathrm{d}\Psi \neq 0$ throughout $\mathscr{D}_{f0}$, then function $\Omega(\Psi)$ can be converted:

$$\Psi = \Psi(\Omega).$$

In terms of $\Psi$, boundary conditions (7.7*b*) become

$$\Psi = \text{const} \quad \text{at} \quad \partial\mathscr{D}_{b0} \quad \text{and} \quad \partial\mathscr{D}. \tag{7.11}$$

Note that, in view of (7.10*a*), (7.11), we have

$$\Omega(\boldsymbol{x}) = \Omega_0 \quad \text{at } \partial\mathscr{D} \quad \text{and} \quad \Omega(\boldsymbol{x}) = \Omega_b \quad \text{at } \partial\mathscr{D}_{b0}$$

with some constants $\Omega_0$ and $\Omega_b$.

### 7.2. *Variational principle*

In this subsection we shall establish an energy-type variational principle for the steady state (7.6). This variational principle is more general than the principles considered in §§ 3–6 in the sense that here we waive the 'isovorticity condition' and consider arbitrary variations of the velocity field $\boldsymbol{u}$. As a consequence it is not just a two-dimensional reduction of the principle of § 3. Moreover, as we shall show at the end of this section, it implies that the two-dimensional version of the principle of § 3 is a particular case corresponding to 'isovortical' variations.

Consider the functional

$$\mathscr{I} = E + \mathscr{C} + A_0\Gamma_0 + A_b\Gamma_b, \tag{7.12}$$

where $A_0$, $A_b$ are arbitrary constants. The functional $\mathscr{I}$ is a linear combination of the invariants (7.3)–(7.5), and therefore $\mathscr{I}$ = const for any solution of the problem (2.1), (7.1), (7.2).

Following our variational procedure of § 3 we introduce the family of transformations

$$\left.\begin{aligned} \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}}(\boldsymbol{x}, \epsilon), \quad \tilde{\boldsymbol{r}} = \tilde{\boldsymbol{r}}(\epsilon), \quad \tilde{\phi} = \tilde{\phi}(\epsilon), \\ \tilde{\boldsymbol{x}}(\boldsymbol{x}, 0) = \boldsymbol{x}, \quad \tilde{\boldsymbol{r}}(0) = 0, \quad \tilde{\phi}(0) = 0, \end{aligned}\right\} \tag{7.13}$$

which is interpreted as a 'virtual motion' of the system. Recall that under this transformation the domain $\mathscr{D}_{f0} = \tilde{\mathscr{D}}_f(0)$ evolves to a new one $\tilde{\mathscr{D}}_f(\epsilon)$ which is entirely determined by the position and the orientation of the rigid body, i.e. by $\tilde{\boldsymbol{r}}(\epsilon)$ and $\tilde{\phi}(\epsilon)$. The variation with 'time' $\epsilon$ of the functions $\tilde{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)$, $\tilde{\boldsymbol{r}}(\epsilon)$ and $\tilde{\phi}(\epsilon)$ is given by the equations

$$\mathrm{d}\tilde{\boldsymbol{x}}/\mathrm{d}\epsilon = \boldsymbol{f}(\tilde{\boldsymbol{x}}, \epsilon), \quad \mathrm{d}\tilde{\boldsymbol{r}}/\mathrm{d}\epsilon = \boldsymbol{h}(\epsilon), \quad \mathrm{d}\tilde{\phi}/\mathrm{d}\epsilon = g(\epsilon), \tag{7.14}$$

where $\boldsymbol{f}(\tilde{\boldsymbol{x}}, \epsilon)$, $\boldsymbol{h}(\epsilon)$ and $g(\epsilon)$ are arbitrary differentiable (with respect to $\epsilon$) functions except that $\boldsymbol{f}(\tilde{\boldsymbol{x}}, \epsilon)$ satisfies the conditions

$$\tilde{\nabla} \cdot \boldsymbol{f} = 0 \quad \text{in} \quad \tilde{\mathscr{D}}_f(\epsilon), \tag{7.15a}$$

$$\boldsymbol{f} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\tilde{\mathscr{D}}. \tag{7.15b}$$

$$\boldsymbol{f} \cdot \boldsymbol{n} = (\boldsymbol{h} + g[\boldsymbol{e}_z \times (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}})]) \cdot \boldsymbol{n} \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(\epsilon). \tag{7.15c}$$

The 'evolution' with $\epsilon$ of the velocity field of the fluid and the velocity of the body is described by functions $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \epsilon)$, $\tilde{\boldsymbol{w}}(\epsilon)$ and $\tilde{\sigma}(\epsilon)$ such that

$$\begin{aligned} &\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, 0) = \boldsymbol{U}(\tilde{\boldsymbol{x}}), \quad \tilde{\boldsymbol{w}}(0) = 0, \quad \tilde{\sigma}(0) = 0, \\ &\tilde{\nabla} \cdot \tilde{\boldsymbol{u}} = 0 \quad \text{in} \quad \tilde{\mathscr{D}}_f, \\ &\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\tilde{\mathscr{D}}, \\ &\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = (\tilde{\boldsymbol{w}} + \tilde{\sigma}[\boldsymbol{e}_z \times (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}})]) \cdot \boldsymbol{n} \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(\epsilon). \end{aligned}$$

The only difference between the variational principle that follows and the principles of §§ 3 and 5 is that we admit arbitrary variations of the velocity field (not only isovortical ones as in §§ 3 and 5 (see (3.10)), i.e.

$$\tilde{\boldsymbol{u}}_\epsilon = \boldsymbol{v}(\tilde{\boldsymbol{x}}, \epsilon), \quad \tilde{\boldsymbol{w}}_\epsilon = \boldsymbol{h}^*(\epsilon), \quad \tilde{\sigma}_\epsilon = g^*(\epsilon) \tag{7.16}$$

with arbitrary functions $\boldsymbol{v}(\tilde{\boldsymbol{x}}, \epsilon)$, $\boldsymbol{h}^*(\epsilon)$ and $g^*(\epsilon)$.

Let us now calculate the first variation of $\mathscr{I}$. Using the formula for differentiating a material volume integral (see Batchelor 1967) we find

$$\delta(E_f + \mathscr{C}) = \int_{\mathscr{D}_{f0}} \rho(\boldsymbol{U} \cdot \delta\boldsymbol{u} + F'(\Omega)\delta\omega)\mathrm{d}\tau + \int_{\partial\mathscr{D}_{b0}} \rho(G + F(\Omega))(\delta\boldsymbol{y} \cdot \boldsymbol{n})\mathrm{d}l. \tag{7.17}$$

Obviously,

$$\delta\Gamma = \int_{\partial\mathscr{D}} \delta\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l}. \tag{7.18}$$

Further, using the formula for differentiating a line integral over material curve (see Batchelor 1967) we obtain

$$\delta\Gamma = \int_{\partial\mathscr{D}_{b0}} \boldsymbol{\tau} \cdot \delta\boldsymbol{u}\,\mathrm{d}l + \int_{\partial\mathscr{D}_{b0}} (\boldsymbol{\tau} \cdot (\delta\boldsymbol{x} \cdot \nabla)\boldsymbol{U} + \boldsymbol{U} \cdot (\boldsymbol{\tau} \cdot \nabla)\delta\boldsymbol{x})\mathrm{d}l. \tag{7.19}$$

In (7.19) $\boldsymbol{\tau} = \boldsymbol{e}_z \times \boldsymbol{n}$ is the unit vector tangent to the curve $\partial\mathscr{D}_{b0}$, so that $\mathrm{d}\boldsymbol{l} = \boldsymbol{\tau}\mathrm{d}l$. It can be shown that

$$\boldsymbol{\tau}\cdot(\delta\boldsymbol{x}\cdot\nabla)\boldsymbol{U}=(\delta\boldsymbol{x}\cdot\boldsymbol{n})\Omega+(\boldsymbol{\tau}\cdot\nabla)(\boldsymbol{U}\cdot\delta\boldsymbol{x})-\boldsymbol{U}\cdot(\boldsymbol{\tau}\cdot\nabla)\delta\boldsymbol{x}.$$

On substituting this in (7.19) and using the fact that $\Omega=\text{const}$ on $\partial\mathscr{D}_{b0}$, we find

$$\delta\Gamma=\int_{\partial\mathscr{D}_{b0}}\delta\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}.$$

Finally, for the first variation of $\mathscr{I}$ we obtain the formula

$$\delta\mathscr{I}=\int_{\mathscr{D}_{f0}}\rho(\boldsymbol{U}\cdot\delta\boldsymbol{u}+F'(\Omega)\delta\omega)\mathrm{d}\tau+\int_{\partial\mathscr{D}_{b0}}\rho(G+F(\Omega))\boldsymbol{n}\cdot\delta\boldsymbol{y}\,\mathrm{d}l$$

$$+M\boldsymbol{W}\cdot\delta\boldsymbol{w}+I\Sigma\delta\sigma+A_0\int_{\partial\mathscr{D}}\delta\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}+A_b\int_{\partial\mathscr{D}_{b0}}\delta\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l},$$

where $F'(\Omega)\equiv\mathrm{d}F(\Omega)/\mathrm{d}\Omega$, and $G$ is given by (7.10*b*). On integrating by parts, $\delta\mathscr{I}$ can be rewritten in the form

$$\delta\mathscr{I}=\int_{\mathscr{D}_{f0}}\rho(\boldsymbol{U}+\nabla\times[F'(\Omega)\boldsymbol{e}_z])\cdot\delta\boldsymbol{u}\,\mathrm{d}\tau+M\boldsymbol{W}\cdot\delta\boldsymbol{w}+I\Sigma\delta\sigma$$

$$+\left(\partial\Pi/\partial r_i+\int_{\partial\mathscr{D}_{b0}}\rho n_i G\mathrm{d}l\right)\delta r_i+\left(\partial\Pi/\partial\phi+\int_{\partial\mathscr{D}_{b0}}\rho\boldsymbol{e}_z\cdot(\boldsymbol{x}\times\boldsymbol{n})G\mathrm{d}l\right)\delta\phi$$

$$+(A_0+\rho F'(\Omega_0))\int_{\partial\mathscr{D}_{b0}}\delta\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}+(A_b+\rho F'(\Omega_b))\int_{\partial\mathscr{D}_{b0}}\delta\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}. \quad (7.20)$$

In (7.20), the derivatives $\partial\Pi/\partial r_i$, $\partial\Pi/\partial\phi$ are taken at $\boldsymbol{r}=0$, $\phi=0$. We now show that with an appropriate choice of the function $F(\omega)$ and constants $A_b$, $A_0$

$$\delta\mathscr{I}=0$$

for any equilibrium given by (7.6)–(7.8).

From (7.20) it is evident that the first variation vanishes under the following conditions:

$$\boldsymbol{W}=0,\quad \Sigma=0; \quad (7.21a)$$

$$A_0=-\rho F'(\Omega_0),\quad A_b=-\rho F'(\Omega_b); \quad (7.21b)$$

$$\boldsymbol{U}=-\nabla\times[F'(\Omega)\boldsymbol{e}_z]; \quad (7.21c)$$

$$\partial\Pi/\partial R_i=-\int_{\partial\mathscr{D}_{b0}}\rho n_i G\mathrm{d}l,\quad \partial\Pi/\partial\phi=-\int_{\partial\mathscr{D}_{b0}}\rho\boldsymbol{e}_z\cdot(\boldsymbol{x}\times\boldsymbol{n})G\,\mathrm{d}l. \quad (7.21d)$$

Note first that (7.21*a*) coincide with (7.6*a*), and therefore they are satisfied. Recalling that constants $A_b$, $A_0$ are still arbitrary we now choose them so as to satisfy the conditions (7.21*b*), in other words, we consider (7.21*b*) as the definitions of $A_b$, $A_0$. Further, in view of (7.9) a natural choice of function $F(\Omega)$ to satisfy (7.21*c*) is given by

$$F'(\Omega)=\Psi(\Omega). \quad (7.22)$$

Finally, according to (7.10*b*), (7.10*c*), (7.11), $G=-P/\rho+\text{const}$ on $\partial\tau_{b0}$, and therefore the conditions (7.21*d*) coincide with (7.8).

Thus, we have shown that *the functional* $\mathscr{I}$ (7.12) *with function* $F(\omega)$ *and constants*

$A_b$, $A_0$ *defined by* (7.22) *and* (7.21*b*) *respectively has a stationary point at any steady solution* (7.6) *of the problem* (2.1), (7.1), (7.2).

This result is a natural generalization of the well-known variational principle obtained by Arnold (1965*a*).

### 7.3. *The second variation*

Calculation of the second variation of $\mathscr{I}$ is very similar to what we have already calculated in §4. In the two-dimensional case the formula (4.4) takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\int_{\tilde{\mathscr{D}}_f} F(\tilde{\boldsymbol{x}},\epsilon)\mathrm{d}\tau = \int_{\tilde{\mathscr{D}}_f} F_{\epsilon\epsilon}\mathrm{d}\tau + \int_{\partial\tilde{\mathscr{D}}_b}(2F_\epsilon + \boldsymbol{y}_\epsilon\cdot\tilde{\nabla}F)(\boldsymbol{y}_\epsilon\cdot\boldsymbol{n})\mathrm{d}s$$
$$+\int_{\partial\tilde{\mathscr{D}}_b} F(\boldsymbol{\tau}\cdot\tilde{\boldsymbol{r}}_\epsilon)\tilde{\phi}_\epsilon\,\mathrm{d}s + \int_{\partial\tilde{\mathscr{D}}_b} F(\tilde{\boldsymbol{r}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}[\boldsymbol{e}_z\times(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{r}})])\cdot\boldsymbol{n}\,\mathrm{d}s.$$

After applying this for calculating $\mathrm{d}^2E_f/\mathrm{d}\epsilon^2$ and after subsequent manipulations that are nearly the same as in §4, it can be shown that

$$\delta^2\mathscr{I} = \delta^2\mathscr{I}_A + \delta^2\mathscr{I}_c + \delta^2\mathscr{I}_b, \tag{7.23a}$$

$$2\delta^2\mathscr{I}_A \equiv \int_{\mathscr{D}_{f0}}\rho((\delta\boldsymbol{u})^2 + \frac{\mathrm{d}\Psi}{\mathrm{d}\Omega}(\delta\omega)^2)\mathrm{d}\tau, \tag{7.23b}$$

$$2\delta^2\mathscr{I}_b \equiv M(\delta\boldsymbol{w})^2 + I(\delta\sigma)^2 + 2\delta^2\Pi, \tag{7.23c}$$

$$2\delta^2\Pi \equiv \Pi_{ik}\delta r_i\delta r_k + 2\Pi_{\phi i}\delta\phi\delta r_i + \Pi_{\phi\phi}(\delta\phi)^2, \tag{7.23d}$$

$$\Pi_{ik} \equiv \left.\frac{\partial^2\Pi}{\partial r_i\partial r_k}\right|_{r,\phi=0},\ \Pi_{\phi i} \equiv \left.\frac{\partial^2\Pi}{\partial\phi\partial r_i}\right|_{r,\phi=0},\ \Pi_{\phi\phi} \equiv \left.\frac{\partial^2\Pi}{\partial\phi^2}\right|_{r,\phi=0}, \tag{7.23e}$$

$$2\delta^2\mathscr{I}_c \equiv \int_{\partial\mathscr{D}_{b0}}\rho(\delta\boldsymbol{y}\cdot\boldsymbol{n})(2\boldsymbol{U}\cdot\delta\boldsymbol{u} + \delta\boldsymbol{y}\cdot\nabla G)\mathrm{d}l + \int_{\partial\mathscr{D}_{b0}}\rho G\delta\phi\delta\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{l}. \tag{7.23f}$$

$\delta^2\mathscr{I}_A$ is precisely the second variation of Arnold (1965*a*) obtained for the problem with a fixed rigid boundary. $\delta^2\mathscr{I}_b$ represents the second variation of the energy of the rigid body alone. And $\delta^2\mathscr{I}_c$ involves the variations of the velocity field of the fluid as well as the variations related to the body and corresponds to the interaction between the fluid flow and the body.

Again, the general theory (Arnold 1965*a*, *b*, 1966; see also Holm *et al.* 1985; Vladimirov 1987*b*) states that if we consider the independent variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{r}$, $\delta\varphi$ as the infinitesimal disturbances to the basic state (7.6), whose evolution is governed by the appropriate linearized equations, then $\delta^2\mathscr{I}$ is an invariant of these equations. If $\delta^2\mathscr{I}$ is positive definite then we can take $\delta^2\mathscr{I}$ as a norm of the disturbance, and the linear stability (in the sense of Lyapunov) of the basic state (7.6) immediately follows from conservation of $\delta^2\mathscr{I}$. We may thus conclude that the stability problem reduces to the problem of identifying the conditions under which $\delta^2\mathscr{I}$ is positive definite.

### 7.4. *'Isovortical' perturbations and flows with constant vorticity*

$\delta^2\mathscr{I}_A$ exists only if $\mathrm{d}\Psi/\mathrm{d}\Omega$ is a bounded function of $\boldsymbol{x}$ in $\mathscr{D}_{f0}$. This evidently is not true if $\Omega' \equiv \mathrm{d}\Omega/\mathrm{d}\Psi$ vanishes at some point in $\mathscr{D}_{f0}$. To deal with such situations we, as in §§3–6, consider 'isovortical' perturbations.

From (3.9) we find that

$$\delta\omega = -\delta\boldsymbol{x}\cdot\nabla\Omega = -\Omega'(\Psi)\delta\boldsymbol{x}\cdot\nabla\Psi, \tag{7.24}$$

where, as before, $\delta\boldsymbol{x}$ is the Lagrangian displacement of a fluid element. In view of (7.24), for ‘isovortical’ perturbations $\delta^2\mathscr{I}_A$ transforms to

$$2\delta^2\mathscr{I}_A \equiv \int_{\mathscr{D}_{f0}} \rho[(\delta\boldsymbol{u})^2 + \Omega'(\Psi)(\delta\boldsymbol{x}\cdot\nabla\Psi)^2]\mathrm{d}\tau. \tag{7.25}$$

For the important class of steady flows with constant vorticity, $\Omega' \equiv 0$, and (7.25) reduces to

$$2\delta^2\mathscr{I}_A \equiv \int_{\mathscr{D}_{f0}} \rho\left(\delta\boldsymbol{u}\right)^2\mathrm{d}\tau,$$

where, according to (7.24), the velocity variation is potential ($\delta\boldsymbol{u} = \nabla\varphi$).

*Remark.* In the two-dimensional problem the variational principles for the states that are steady in a reference frame translationally moving or rotating round some fixed axis can be established by considering the linear combinations of the energy and the momentum or of the energy and the angular momentum of the system. In a particular case of ‘isovortical’ perturbations the corresponding second variations are obtained by the two-dimensional reduction of the expressions (5.22) and (6.12).

## 8. Applications

As was demonstrated by Rouchon (1991) and by Sadun & Vishik (1993), in the three-dimensional case Arnold’s second variation of the kinetic energy is, in general, indefinite in sign. $\delta^2E_A$, entering the general formula (4..11) for the second variation, is the same as Arnold’s second variation (Arnold 1965*b*) and, therefore, is also indefinite in sign. Clearly, $\delta\boldsymbol{u}$, $\delta\boldsymbol{x}$ and $\delta\boldsymbol{\omega}$ defined in the domain $\mathscr{D}_{b0}$ may be prescribed independently of the variations in the position and orientation of the body. Therefore, possible sign-indefiniteness of $\delta^2E_A$ cannot be compensated by $\delta^2E_c$ and $\delta^2E_b$ also entering the expression (4.11). Thus, we may conclude that in the three-dimensional case most of the steady states of the system ‘body + fluid’ correspond to sign-indefinite second variations. Nevertheless, there is a number of exceptional situations (such as rigid rotation, potential flows, flows with constant vorticity and two-dimensional flows †) which are interesting and which can give us positive-definite second variations and corresponding stability criteria. To demonstrate this, we shall apply the general theory of the preceding sections to three particular stability problems.

### 8.1. *Steady translational motion of a cylinder* (*irrotational flow*)

Consider an infinitely long cylinder of an arbitrary cross-section steadily moving perpendicular to its axis through a fluid that extends to infinity in all directions and is at rest there. We assume that the flow is irrotational and two-dimensional, and that there is non-zero circulation of velocity about the cylinder. Homogeneous gravitational force is also taken into consideration.

In the basic state whose stability is studied the cylinder moves along one of the principal axes of its added-mass tensor (it can be easily shown that this is the only

† Some results on the two-dimensional problem may be found in the papers by Vladimirov & Ilin (1994, 1996, 1998).

possible purely translational motion). Let it be the $x$-axis. The basic state is given by

$$\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x} - \boldsymbol{R}(t)) = \nabla\varphi_0, \quad \boldsymbol{w} = \dot{\boldsymbol{R}} = U_0\boldsymbol{e}_x, \quad \boldsymbol{R} = U_0 t\boldsymbol{e}_x, \quad \sigma = 0, \tag{8.1}$$

with the velocity potential $\varphi_0(\boldsymbol{x} - \boldsymbol{R}(t))$ being a solution of the following problem:

$$\nabla^2\varphi_0 = 0 \quad \text{in} \ \mathscr{D}_f, \quad \boldsymbol{n}\cdot\nabla\varphi_0 = \boldsymbol{n}\cdot\boldsymbol{e}_x U_0 \quad \text{on} \ \partial\mathscr{D}_{b0}, \quad \int_{\partial\mathscr{D}_{b0}} \nabla\varphi_0\cdot \mathrm{d}\boldsymbol{l} = \Gamma. \tag{8.2}$$

The pressure is determined from Bernoulli's equation

$$P/\rho - U_0\boldsymbol{e}_x\cdot\nabla\varphi_0 + \left(\nabla\varphi_0\right)^2/2 + g\boldsymbol{e}_y\cdot\boldsymbol{x} = 0.$$

The balance of the forces on the body reduces to

$$\rho\Gamma U_0 = (M-\mu)g, \quad \mu \equiv \int_{\partial\mathscr{D}_{b0}} \rho\mathrm{d}\tau. \tag{8.3}$$

We assume that the body is homogeneous in density, i.e. its geometrical centre coincides with the centre of mass:

$$\int_{\mathscr{D}_b} (\boldsymbol{x} - \boldsymbol{R})\mathrm{d}\tau = 0. \tag{8.4}$$

One more assumption is that the conformal centre of the body (see e.g. Milne-Thomson 1973) coincides with its geometrical centre†. It may be shown that this, in view of (8.4), has a consequence that

$$\int_{\partial\mathscr{D}_{b0}} (\boldsymbol{x} - \boldsymbol{R})\,\nabla\varphi^{\Gamma}\cdot\mathrm{d}\boldsymbol{l} = 0 \tag{8.5}$$

where $\varphi^{\Gamma}$ is the cyclic part of the potential $\varphi_0$ and is defined as a solution of the problem

$$\nabla^2\varphi^{\Gamma} = 0 \quad \text{in} \ \mathscr{D}_{f0}, \quad \boldsymbol{n}\cdot\nabla\varphi^{\Gamma} = 0 \quad \text{on} \ \partial\mathscr{D}_{b0}, \quad \int_{\partial\mathscr{D}_{b0}} \nabla\varphi^{\Gamma}\cdot\mathrm{d}\boldsymbol{l} = \Gamma. \tag{8.6}$$

Equations (8.4), (8.5) and the assumption that the orientation of the body is such that one of the principal axes of its added-mass tensor is parallel to the $x$-axis have, in turn, a consequence that the total torque (relative to the centre of mass) exerted on the body by the fluid vanishes, i.e.

$$\int_{\partial\mathscr{D}_{b0}} \boldsymbol{e}_z\cdot\left[(\boldsymbol{x} - \boldsymbol{R})\times\boldsymbol{n}\right]P\,\mathrm{d}l = 0. \tag{8.7}$$

*Variational principle*. When the circulation about the body is non-zero the energy of the fluid (as well as the momentum) is infinite. In this situation, as was mentioned in § 5 it is natural to use the vortex energy and vortex momentum. Then the total energy and the $x$-component of the total momentum of the system are given by

$$E = E_f + E_b, \quad E_f \equiv -\tfrac{1}{2}\int_{\partial\mathscr{D}_b} \rho\psi\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}, \quad E_b \equiv \tfrac{1}{2}M\boldsymbol{w}^2 + \tfrac{1}{2}I\sigma^2 + (M-\mu)g\boldsymbol{e}_y\cdot\boldsymbol{r},$$

$$N = N_f + N_b, \quad N_f = \int_{\partial\mathscr{D}_b} \rho(\boldsymbol{e}_y\cdot\boldsymbol{x})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}, \quad N_b = M\dot{\boldsymbol{r}}.$$

† It is definitely so at least for the bodies that are symmetric relative to both coordinate axes, e.g. for an ellipse.

Conservation of $E$ and $N$ given by these formulae is easily verified by direct calculation. We shall show that the conserved functional $\mathscr{I} = E - U_0 N$ has a critical point in the basic state (8.1).

For an irrotational flow, (5.17) reduces to

$$\tilde{\boldsymbol{u}}_\epsilon = \tilde{\nabla}\alpha, \tag{8.8}$$

where function $\alpha(\tilde{\boldsymbol{x}}, t, \epsilon)$ is a solution of the problem (cf. (5.18))

$$\tilde{\nabla}^2\alpha = 0 \quad \text{in} \quad \tilde{\mathscr{D}}_f(t,\epsilon), \quad \boldsymbol{n}\cdot\tilde{\nabla}\alpha = \boldsymbol{n}\cdot\tilde{\boldsymbol{u}}_\epsilon \quad \text{on} \quad \partial\tilde{\mathscr{D}}_b(t,\epsilon), \quad \int_{\partial\tilde{\mathscr{D}}_b(t,\epsilon)} \tilde{\nabla}\alpha\cdot \mathrm{d}\boldsymbol{l} = 0. \tag{8.9}$$

We suppose that the value of $\boldsymbol{n}\cdot\tilde{\boldsymbol{u}}_\epsilon$ on $\tilde{\mathscr{D}}_b(t,\epsilon)$ has been determined by differentiation (with respect to $\epsilon$) of the boundary condition (5.15*b*).

Consider now the functional $\mathscr{I} = E - U_0 N$. Differentiation of $E$ and $N$ with respect to $\epsilon$ at $\epsilon = 0$ gives

$$\delta E = -\int_{\partial\mathscr{D}_{b0}} \rho[(\delta\boldsymbol{u}\cdot\boldsymbol{\tau})\Psi - (\delta\boldsymbol{y}\cdot\boldsymbol{n})\boldsymbol{U}^2/2]\mathrm{d}l + MU_0\boldsymbol{e}_x\cdot\delta\boldsymbol{w} + (M-\mu)g\boldsymbol{e}_y\cdot\delta\boldsymbol{r} \tag{8.10}$$

and

$$\delta N = \int_{\partial\mathscr{D}_{b0}} \rho[(\delta\boldsymbol{u}\cdot\boldsymbol{\tau})\,\boldsymbol{x}\cdot\boldsymbol{e}_y + (\delta\boldsymbol{y}\cdot\boldsymbol{n})\,\boldsymbol{e}_x\cdot\nabla\varphi_0]\mathrm{d}l + M\delta\boldsymbol{w}. \tag{8.11}$$

Since

$$\begin{aligned}\int_{\partial\mathscr{D}_{b0}} \rho(\delta\boldsymbol{y}\cdot\boldsymbol{n})\boldsymbol{U}^2/2\,\mathrm{d}l &= \int_{\partial\mathscr{D}_{b0}} (\delta\boldsymbol{y}\cdot\boldsymbol{n})(\rho U_0\boldsymbol{e}_x\cdot\nabla\varphi_0 - P - \rho g\boldsymbol{e}_y\cdot\boldsymbol{x})\mathrm{d}l \\ &= \int_{\partial\mathscr{D}_{b0}} (\delta\boldsymbol{y}\cdot\boldsymbol{n})\rho U_0\boldsymbol{e}_x\cdot\nabla\varphi_0\mathrm{d}l - \int_{\partial\mathscr{D}_{b0}} (\delta\boldsymbol{y}\cdot\boldsymbol{n})P\,\mathrm{d}l + \mu g\boldsymbol{e}_y\cdot\delta\boldsymbol{r},\end{aligned}$$

we obtain

$$\begin{aligned}\delta\mathscr{I} = &-\int_{\partial\mathscr{D}_{b0}} \rho(\delta\boldsymbol{u}\cdot\boldsymbol{\tau})(\Psi + U_0\boldsymbol{x}\cdot\boldsymbol{e}_y)\mathrm{d}l + \left\{Mg\boldsymbol{e}_y - \int_{\partial\mathscr{D}_{b0}} P\boldsymbol{n}\,\mathrm{d}l\right\}\cdot\delta\boldsymbol{r} \\ &-\int_{\partial\mathscr{D}_{b0}} \delta\phi\boldsymbol{e}_z\cdot[(\boldsymbol{x}-\boldsymbol{R})\times\boldsymbol{n}]P\mathrm{d}l,\end{aligned}$$

whence, in view of (8.3), (8.7),

$$\delta\mathscr{I} = -\int_{\partial\mathscr{D}_{b0}} \rho(\delta\boldsymbol{u}\cdot\boldsymbol{\tau})(\Psi + U_0\boldsymbol{x}\cdot\boldsymbol{e}_y)\mathrm{d}l. \tag{8.12}$$

From the boundary condition $\boldsymbol{n}\cdot\nabla\varphi_0 = \boldsymbol{n}\cdot\boldsymbol{e}_x U_0$ on $\partial\mathscr{D}_{b0}$ and from the definition of the stream function $\Psi$ (7.9), we find that

$$\Psi = -U_0\boldsymbol{x}\cdot\boldsymbol{e}_y + \text{const} \quad \text{on} \quad \partial\mathscr{D}_{b0},$$

and therefore $\delta\mathscr{I} = 0$.

*Stability criterion.* Calculation of the second variation of $\mathscr{I}$ is analogous to calculations of §§ 4–7 and results in (cf. (7.23))

$$\begin{aligned}2\delta^2\mathscr{I} = -\int_{\partial\mathscr{D}_{b0}} \rho[&(\delta\boldsymbol{u}\cdot\boldsymbol{\tau})\delta\psi - (\delta\boldsymbol{y}\cdot\boldsymbol{n})(2\boldsymbol{U}^*\cdot\delta\boldsymbol{u} + (\delta\boldsymbol{y}\cdot\nabla)\boldsymbol{U}^{*2}/2) \\ &+\delta\phi(\delta\boldsymbol{r}\cdot\boldsymbol{\tau})\boldsymbol{U}^{*2}/2)]\mathrm{d}l + M(\delta\dot{\boldsymbol{r}})^2 + I(\delta\dot{\phi})^2.\end{aligned} \tag{8.13}$$

Here $\boldsymbol{U}^* \equiv \nabla\varphi_0 - U_0\boldsymbol{e}_x$. After standard but quite tedious manipulations using the irrotatonal character of the basic state and the boundary condition

$$\boldsymbol{n}\cdot\nabla\delta\varphi = \delta\dot{\boldsymbol{y}}\cdot\boldsymbol{n} + \boldsymbol{\tau}\cdot\nabla\left[\left(\delta\boldsymbol{y}\cdot\boldsymbol{n}\right)\left(\boldsymbol{U}\cdot\boldsymbol{\tau}\right)\right] \quad \text{on } \partial\mathscr{D}_b$$

(which can be derived by differentiating the condition (3.7$c$) with respect to $\epsilon$), formula (8.13) can be transformed to

$$\delta^2\mathscr{I} = \tfrac{1}{2}\int_{\mathscr{D}_{f0}} \rho(\delta\tilde{\boldsymbol{u}})^2 \mathrm{d}\tau + \tfrac{1}{2}M\left(\delta\dot{\boldsymbol{r}}\right)^2 + \tfrac{1}{2}I\left(\delta\dot{\phi}\right)^2$$

$$+\tfrac{1}{2}\left(\mu_{11}-\mu_{22}\right)U_0^2\left(\delta\phi\right)^2 + U_0\left(\delta\phi\right)^2\int_{\partial\mathscr{D}_{b0}} \rho\left[\boldsymbol{e}_y\cdot(\boldsymbol{x}-\boldsymbol{R})\right]\nabla\varphi^{\Gamma}\cdot\mathrm{d}\boldsymbol{l} \quad (8.14)$$

where $\mu_{11}$, $\mu_{22}$ are (main) added-mass coefficients corresponding to a motion of the body along the $x$- and $y$-axes respectively and where $\delta\tilde{\boldsymbol{u}}$ is the velocity field of the fluid produced by a body moving translationally with velocity $\delta\dot{\boldsymbol{r}}$ and rotating with angular velocity $\delta\dot{\phi}$. Comparing the integral over the body surface in (8.14) with (8.5), we conclude that it vanishes, and the second variation reduces to

$$\delta^2\mathscr{I} = \tfrac{1}{2}\int_{\mathscr{D}_{f0}} \rho(\delta\tilde{\boldsymbol{u}})^2 \mathrm{d}\tau + \tfrac{1}{2}M\left(\delta\dot{\boldsymbol{r}}\right)^2 + \tfrac{1}{2}I\left(\delta\dot{\phi}\right)^2 + \tfrac{1}{2}\left(\mu_{11}-\mu_{22}\right)U_0^2\left(\delta\phi\right)^2. \quad (8.15)$$

Now it is clear that $\delta^2\mathscr{I}$ is positive definite and therefore the basic state (8.1) is linearly stable provided that $\mu_{11} > \mu_{22}$. We have thus obtained the following stability criterion: *the translational motion of a rigid body along the principal axis of its added-mass tensor which corresponds to a maximum added mass is linearly stable.*

Note that for a translational motion of a body without circulation ($\Gamma = 0$) and without the external gravity force, the corresponding second variation and the stability condition are the same as above. We therefore may conclude that the presence of non-zero circulation does not affect the stability of the body (at least for the class of bodies considered here) – an interesting result which seems to be new.

*Remark*: *a circular cylinder in a flow with constant vorticity*. The result obtained above shows that possible instability of a body may result in rotation of the body around its centre of mass, and the instability may happen only if the body is not a circular cylinder (translational motion of a circular cylinder is stable). This, however, is not true if the flow is not irrotational. To illustrate this, we consider the stability of a circular cylinder translationally moving through a fluid which is in simple shearing flow at infinity. In a reference frame moving with the cylinder, the basic state is given by the equations (see Batchelor 1967)

$$\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}) = -\nabla\times(\Psi\boldsymbol{e}_z), \quad \boldsymbol{w} = \dot{\boldsymbol{R}} = 0, \quad \boldsymbol{R} = 0, \quad \sigma = 0;$$

$$\Psi = \tfrac{1}{2}\Omega_0 r^2\sin^2\theta + U_0 r\sin\theta + \frac{\Gamma}{2\pi}\log r - \frac{1}{r}U_0 a^2\sin\theta + \frac{1}{4r^2}\Omega_0 a^4\cos 2\theta.$$

Here $a$ is the radius of the cylinder, $(r,\theta)$ are polar coordinates, and $\boldsymbol{U}(\boldsymbol{x}) \to -\boldsymbol{e}_x(U_0 + \Omega_0 y)$ as $r \to 0$. In the basic state, the gravitational force is balanced by the lift force:

$$\rho U_0\left(2\pi a^2\Omega_0 + \Gamma\right) = (M-\mu)g, \quad \mu \equiv \int_{\partial\mathscr{D}_{b0}} \rho\mathrm{d}\tau.$$

It may be shown that the second variation (7.23) reduces to

$$2\delta^2 E = (M+\mu)(\delta\dot{\boldsymbol{r}})^2 - (2\mu\Omega_0 + \rho\Gamma)\Omega_0(\delta r_2)^2, \quad \mu = \rho\pi a^2.$$

Evidently, in contrast to the case of irrotational flow, there are situations when $\delta^2 E$ is indefinite in sign (for example, if $U_0 > 0$, $\Omega_0 > 0$ and $M - \mu > 0$).

### 8.2. *Steady rotation of a force-free rigid body with a fluid-filled cavity*

Consider now the force-free motion of a rigid body with a cavity completely filled with an ideal fluid. Obviously, the only non-trivial motion of such a system is motion relative to a fixed centre of mass of the whole system. The basic state whose stability will be studied is the rotation with constant angular velocity $\sigma_0$ of the whole system round the $z$-axis passing through the centre of mass of the system. For this basic state, equations (6.7), (6.8) can be shown to have a consequence that the $z$-axis must be one of the principle axes of the moment of inertia of the system, i.e.

$$I^s_{13} = I^s_{23} = 0, \quad I^s_{ik} = I_{ik} + I^f_{ik}, \quad I^f_{ik} \equiv \int_{\mathscr{D}_{f0}} \rho\left(\boldsymbol{x}^2\delta_{ik} - x_i x_k\right)\mathrm{d}\tau.$$

Without loss of generality, the $(x', y')$-axes fixed in the body can be taken so as to coincide with the principal axes of inertia of the system; the inertia tensor $\hat{\boldsymbol{I}}^s$ relative to a set of axes fixed in space is then given by the formula $\hat{\boldsymbol{I}}^s = \hat{\boldsymbol{P}}\hat{\boldsymbol{I}}^*\hat{\boldsymbol{P}}^{-1}$ where the matrix $\hat{\boldsymbol{P}}$ is given by (6.5*b*) and the inertia tensor of the system written in the body set of axes has a diagonal form: $\hat{\boldsymbol{I}}^* = \mathrm{diag}\{I^*_{11}, I^*_{22}, I^*_{33}\}$ (obviously, $I^*_{33} = I^s_{33}$). Then after tedious but elementary manipulations, the second variation (6.12) can be reduced to the expression

$$\delta^2\mathscr{I} = \tfrac{1}{2}\int_{\mathscr{D}_{f0}} \rho(\delta\boldsymbol{u})^2 + \tfrac{1}{2}\delta\boldsymbol{\sigma}\cdot\hat{\boldsymbol{I}}^s\cdot\delta\boldsymbol{\sigma} + \tfrac{1}{2}M(\delta\boldsymbol{\sigma}\times\boldsymbol{r} - \sigma_0[\boldsymbol{e}_z\times[\boldsymbol{e}_z\times\delta\boldsymbol{r}]])^2$$

$$+\tfrac{1}{2}\sigma_0^2(I^*_{33} - I^*_{22})(\delta\psi)^2 + \tfrac{1}{2}\sigma_0^2(I^*_{33} - I^*_{11})(\delta\theta)^2.$$

One can see that $\delta^2\mathscr{I}$ is positive definite and therefore the steady rotation of the system is linearly stable if $I^*_{33} > I^*_{11}$ and $I^*_{33} > I^*_{22}$, i.e. the system rotates as a whole round the principle axis of its moment of inertia tensor which corresponds to the greatest value of the moment of inertia. Thus, we have rediscovered the result first obtained by Rumyantsev (see Moiseev & Rumyantsev 1965). We should mention here that in fact the same condition provides the stability to arbitrary finite-amplitude perturbations and also remains valid in the case of a viscous fluid (Moffatt & Ilin 1994).

### 8.3. *Steady rotation of a body in a rotating fluid*

Consider now a problem which is complementary to the previous one: the stability of a rigid body placed in a rotating fluid inside an axisymmetric domain. In the basic state, both the body and the fluid rotate with constant angular velocity $\sigma_0$ around the symmetry axis (say, the z-axis) of the domain. The flow is steady relative to the frame of reference rotating with the same angular velocity and, as was mentioned in the end of § 6, the theory of § 6 is applicable.

We assume that the body is homogeneous in density and, therefore, its centre of mass coincides with its geometrical centre. Then, it may be shown that the equations of the balance (6.8) are satisfied without any external force and torque ($\Pi = 0$) provided that one of the principal axes of body's moment of inertia tensor coincides with the axis of the symmetry of the domain. This means that $I_{31} = I_{32} = 0$.

It may be shown by standard manipulations that for the problem considered the second variation (6.12) reduces to

$$\delta^2\mathscr{I} = \tfrac{1}{2}\int_{\mathscr{D}_{f0}} \rho(\delta\boldsymbol{u})^2 + \tfrac{1}{2}\delta\boldsymbol{\sigma}\cdot\hat{\boldsymbol{I}}\cdot\delta\boldsymbol{\sigma} + \tfrac{1}{2}M(\delta\dot{\boldsymbol{r}} - \sigma_0\boldsymbol{e}_z\times\delta\boldsymbol{r})^2$$

$$+\tfrac{1}{2}\mu\sigma_0^2\left(1-\frac{M}{\mu}\right)(\boldsymbol{e}_z\times\delta\boldsymbol{r})^2 + \tfrac{1}{2}\sigma_0^2\left(1-\frac{\mu}{M}\right)((I_{33}^b - I_{22}^b)(\delta\psi)^2 + (I_{33}^b - I_{11}^b)(\delta\theta)^2),$$

where $I_{11}^b$, $I_{22}^b$ and $I_{33}^b$ are the principal moments of inertia of the body and, as before (cf. (8.3)), $\mu$ is the mass of the fluid displaced by the body.

Evidently, $\delta^2\mathscr{I}$ is positive definite and, therefore, the basic state is linearly stable if (i) $M < \mu$, i.e. the density of the fluid is greater than that of the body, and (ii) $I_{33}^b < I_{11}^b$, $I_{33}^b < I_{22}^b$, i.e the axis of rotation corresponds to the smallest moment of inertia of the body. In particular, if the body is an ellipsoid, its rotation is stable provided that it rotates around its longest axis.

## 9. Conclusion

In this paper we have established four energy-type variational principles for general steady states of the dynamical system 'ideal fluid + rigid body'. The system may be a rigid body placed in an arbitrary steady flow of an inviscid incompressible fluid or it may be a body with a cavity entirely filled with a fluid; no restrictions on the form of the body or the cavity were imposed. Variational principles have been constructed for an arbitrary steady state of the system and for unsteady states that are steady either in a translationally moving reference frame or in a rotating (around some fixed axis) frame of reference. In the first case it has been shown that the energy of the system has a stationary value on the set of all states of the system in which fluid flows are 'isovortical' to the basic steady flow. In the second one a certain linear combination of the energy and either the momentum or the angular momentum of the system has a critical point on the same set of 'isovortical' flows.

We have then considered the two-dimensional system 'body + fluid', and have constructed the variational principle for arbitrary steady states of the system. In contrast with the three-dimensional case, here we have admitted arbitrary variations of the velocity field of the fluid (not only 'isovortical' ones).

For all problems considered the second variations of the corresponding functionals have been calculated. The general theory of Arnold (1965*a*, *b*, 1966) states that the steady solution is stable (at least in the linear approximation) if the corresponding functional is of definite sign. Therefore, the stability problem effectively reduces to the analysis of the second variation.

The general theory developed in the paper has been applied to three particular problems where the sufficient conditions for stability were obtained, thus proving the usefulness of the theory.

Many interesting particular problems remain unsolved in this area, in particular the two-dimensional problem of the stability of a body in a flow with constant vorticity. This is a subject of a continuing investigation.

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## Appendix. Derivation of equation (4.4)

It is well known that the rate of change of a material volume integral is given by the formula (see e.g. Batchelor 1967)

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_{\tilde{\mathscr{D}}_f(\epsilon)} F(\tilde{\boldsymbol{x}},\epsilon)\mathrm{d}\tau = \int_{\tilde{\mathscr{D}}_f(\epsilon)} (F_\epsilon + \nabla\cdot(F\boldsymbol{f}))\mathrm{d}\tau \tag{A 1}$$

where $F(\tilde{\boldsymbol{x}},\epsilon)$ is an arbitrary sufficiently smooth function and $F_\epsilon \equiv \partial F/\partial\epsilon$. On using this formula one more time we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\int_{\tilde{\mathscr{D}}_f(\epsilon)} F(\tilde{\boldsymbol{x}},\epsilon)\mathrm{d}\tau = \int_{\tilde{\mathscr{D}}_f(\epsilon)} [F_{\epsilon\epsilon} + \nabla\cdot([2F_\epsilon + \nabla\cdot(F\boldsymbol{f})]\boldsymbol{f}) + \nabla\cdot(F\boldsymbol{f}_\epsilon)]\mathrm{d}\tau. \tag{A 2}$$

To proceed further we need boundary conditions for the function $\boldsymbol{f}_\epsilon(\boldsymbol{x},\epsilon)$. Differentiation with respect to $\epsilon$ of the conditions (3.4$b,c$) results in

$$\boldsymbol{f}_\epsilon\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\tilde{\mathscr{D}}. \tag{A 3a}$$

$$\boldsymbol{f}_\epsilon\cdot\boldsymbol{n} = \boldsymbol{y}_{\epsilon\epsilon}\cdot\boldsymbol{n} + (\boldsymbol{y}_\epsilon - \boldsymbol{f})\cdot(\boldsymbol{\varphi}_\epsilon\times\boldsymbol{n}) - \boldsymbol{n}\cdot(\boldsymbol{y}_\epsilon\cdot\nabla)\boldsymbol{f} \quad \text{on } \partial\tilde{\mathscr{D}}_b(\epsilon), \tag{A 3b}$$

where we have used the obvious formula $\boldsymbol{n}_\epsilon = \boldsymbol{\varphi}_\epsilon\times\boldsymbol{n}$, $\boldsymbol{y}_\epsilon$ is given by (4.5), and where

$$\begin{aligned}\boldsymbol{y}_{\epsilon\epsilon} &= \tilde{\boldsymbol{r}}_{\epsilon\epsilon} + \tilde{\boldsymbol{\varphi}}_{\epsilon\epsilon}\times(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}}) + \tilde{\boldsymbol{\varphi}}_\epsilon\times(\tilde{\boldsymbol{x}}_\epsilon - \tilde{\boldsymbol{r}}_\epsilon)\\ &= \tilde{\boldsymbol{r}}_{\epsilon\epsilon} + \tilde{\boldsymbol{\varphi}}_{\epsilon\epsilon}\times(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}}) + \tilde{\boldsymbol{\varphi}}_\epsilon\times[\tilde{\boldsymbol{\varphi}}_\epsilon\times(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{r}})].\end{aligned} \tag{A 4}$$

Applying now the divergence theorem to the integral in the right-hand side of (A 2) and taking account of the boundary conditions (3.4$b,c$) and (A 3$a,b$), we obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\int_{\tilde{\mathscr{D}}_f(\epsilon)} F(\tilde{\boldsymbol{x}},\epsilon)\mathrm{d}\tau &= \int_{\tilde{\mathscr{D}}_f(\epsilon)} F_{\epsilon\epsilon}\mathrm{d}\tau + \int_{\partial\tilde{\mathscr{D}}_b(\epsilon)} [2F_\epsilon + \boldsymbol{f}\cdot\nabla F](\boldsymbol{y}_\epsilon\cdot\boldsymbol{n})\mathrm{d}s\\ &\quad + \int_{\partial\tilde{\mathscr{D}}_b(\epsilon)} F[\boldsymbol{y}_{\epsilon\epsilon}\cdot\boldsymbol{n} + (\boldsymbol{y}_\epsilon - \boldsymbol{f})\cdot(\boldsymbol{\varphi}_\epsilon\times\boldsymbol{n}) - \boldsymbol{n}\cdot(\boldsymbol{y}_\epsilon\cdot\nabla)\boldsymbol{f}]\mathrm{d}s.\end{aligned} \tag{A 5}$$

Consider now the integral (which appears in (A 5))

$$\mathscr{I}_1 = \int_{\partial\tilde{\mathscr{D}}_b(\epsilon)} [(\boldsymbol{f}\cdot\nabla F)(\boldsymbol{y}_\epsilon\cdot\boldsymbol{n}) - F\boldsymbol{n}\cdot(\boldsymbol{y}_\epsilon\cdot\nabla)\boldsymbol{f}]\mathrm{d}s.$$

Note first that

$$Fn_k y_{i\epsilon}\partial_i f_k = n_k y_{i\epsilon}\partial_i(Ff_k) - n_k f_k y_{i\epsilon}\partial_i F,$$

whence the integrand in $\mathscr{I}_1$ can be written in the form

$$(\boldsymbol{f}\cdot\nabla F)(\boldsymbol{y}_\epsilon\cdot\boldsymbol{n}) - F\boldsymbol{n}\cdot(\boldsymbol{y}_\epsilon\cdot\nabla)\boldsymbol{f} = (\boldsymbol{y}_\epsilon\cdot\nabla F)(\boldsymbol{y}_\epsilon\cdot\boldsymbol{n}) - B$$

where

$$\begin{aligned}B &\equiv y_{i\epsilon}n_k\partial_i(Ff_k) - n_k y_{k\epsilon}f_i\partial_i F\\ &= n_k\partial_i[(y_{i\epsilon}f_k - y_{k\epsilon}f_i)F] - Fn_k f_k\partial_i y_{i\epsilon} + Fn_k f_i\partial_i y_{k\epsilon}.\end{aligned}$$

It follows from (4.5) that

$$\partial_i y_{i\epsilon} = 0, \qquad \partial_i y_{k\epsilon} = e_{ikl}\phi_{l\epsilon}.$$

Therefore,

$$B = \boldsymbol{n}\cdot\nabla\times(F[\boldsymbol{f}\times\boldsymbol{y}_\epsilon]) + F\boldsymbol{f}\cdot[\boldsymbol{n}\times\boldsymbol{\varphi}_\epsilon],$$

and $\mathscr{I}_1$ simplifies to

$$\mathscr{I}_1 = \int_{\partial \tilde{\mathscr{D}}_b(\epsilon)} \left[ \left( \boldsymbol{y}_\epsilon \cdot \nabla F \right) \left( \boldsymbol{y}_\epsilon \cdot \boldsymbol{n} \right) - F \boldsymbol{f} \cdot [\boldsymbol{n} \times \boldsymbol{\varphi}_\epsilon] \right] \mathrm{d}s, \tag{A 6}$$

Finally, after substitution of (A 6) in (A 5) and some simple manipulations with the help of (A 4), we arrive at (4.4).

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